Asimov's GALAXY

DEFORMATIONS OF PLANE RECTILINEAR COMPLEXES*

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1. The deformation theorem. On a euclidean plane, consider two fixed triangles, $A_1A_2A_3$ and $B_1B_2B_3$. Let P_1, P_2, and P_3 be three variable points of the plane. Suppose that $P_i (i=1, 2, 3)$ moves in continuous fashion from A_i to B_i. It is then easily seen that the variable triangle $P_1P_2P_3$ can be required to remain nondegenerate throughout the motion unless one of the cyclic orderings $A_1A_2A_3$ and $B_1B_2B_3$ is counterclockwise and the other is clockwise.

We will consider a more general problem of the same sort. It will be understood throughout that all the geometric configurations are on a euclidean plane. Let K_0 be a collection of triangles, any two of which are distinct or else have in common either a single vertex or two vertices and the edge joining them. Let $(A_1, \cdots, A_n)$ be the set of all points each of which is a vertex of at least one of the triangles K_0. Such a collection of triangles as K_0 is called a (*plane rectilinear simplicial 2-dimensional*) *complex*. Since all the complexes with which we deal will be plane, rectilinear, simplicial, and 2-dimensional, these modifiers will be omitted. A similar convention will apply to all words enclosed in parentheses in our definitions. The vertices, edges, and triangles of K_0 are referred to as the *cells* of the complex, of dimensions 0, 1, and 2 respectively.

Let K_1 be a second complex with vertices $(B_1, \cdots, B_n)$ where $B_iB_jB_k$ is a cell of K_1 when and only when $A_iA_jA_k$ is a cell of K_0. We refer to K_0 and K_1 as being *isomorphic* (*with vertices similarly numbered*).

The Deformation Problem. Is it possible for a set $(P_1, \cdots, P_n)$ *of points to move continuously so that* (1) P_i *goes from* A_i *to* B_i $(i=1, \cdots, n)$ *and* (2) *throughout the motion,* $(P_1, \cdots, P_n)$ *are the vertices of a complex,* K, *isomorphic to* K_0? *Whenever the answer to this question is affirmative, we will say that* K_0 *is deformable into* K_1.

One can visualize the problem as follows. Think of $(A_1, \cdots, A_n)$ as pegs movable in the plane. Suppose that a piece of rubber band joins A_i to A_j when and only when A_iA_j is a cell of K_0; and suppose further that the rubber remains taut no matter how close together A_i and A_j are brought. Is it then possible to move the pegs $(A_1, \cdots, A_n)$ into the respective positions $(B_1, \cdots, B_n)$ without any rubber bands running foul of one another or of the pegs?

It is not difficult to concoct examples of various sorts in which the answer to the question just raised is negative. The object of the present paper is to prove a deformation theorem for an interesting class of complexes sufficiently restricted to avoid the complications of the most general case.†

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* This paper contains the substance of an address entitled "Introduction of a Riemannian Geometry on a Manifold," delivered by the writer at the Summer Meeting of the Association, September 11, 1943, in New Brunswick, New Jersey.

† For a general discussion, directed toward an application to the problem of introducing an analytic Riemannian geometry on a manifold, see S. S. Cairns, Isotopic deformations of geodesic complexes on the 2-sphere and on the plane, to appear in the Annals of Mathematics.

2. A special deformation theorem. Euler's relationship.

THEOREM. *Let K_0 and K_1 be isomorphic complexes with vertices $(A_1, \cdots, A_n)$ and $(B_1, \cdots, B_n)$ respectively. Suppose that* (1) $A_i \equiv B_i$ $(i=1, 2, 3)$ (2) *K_0, and therefore K_1, exactly covers the triangle $A_1A_2A_3$, and* (3) *all vertices save A_1, A_2, and A_3 are interior to the triangle $A_1A_2A_3$. Then K_0 is deformable into K_1. It can be required that A_1, A_2, and A_3 remain fixed during the deformation.*

The most general rectilinear deformation theorem in the plane can be shown to follow from this special theorem, applied to corresponding parts of such complexes as K_0 and K_1.

Our discussion will involve the well-known Euler relationship

$$\alpha_0 - \alpha_1 + \alpha_2 = 1 \tag{2.1}$$

where α_0, α_1, and α_2 are respectively, the numbers of vertices, edges, and triangles belonging to K_0.

By the *star*, $S(A_i)$, of the vertex A_i, we will mean A_i, together with all the 1-dimensional and 2-dimensional cells of K_0 having A_i as a vertex. Let ν_i $(i=1, \cdots, n)$ be the number of 1-dimensional cells belonging to $S(A_i)$. Then

$$\begin{aligned} \sum_{i=1}^{n} \nu_i &= 2\alpha_1 \\ \sum_{i=1}^{n} \nu_i &= 3\alpha_2 + 3. \end{aligned} \tag{2.2}$$

For, (1) in the first summation, each edge can be thought of as counted twice (once for each end-point) and (2) in the second summation, each of the α_2 triangles of K_0, and the exterior of $A_1A_2A_3$, can be thought of as counted three times. Multiplying (2.1) by six, eliminating α_1 and α_2 with the aid of (2.2), and replacing α_0 by n, we deduce

$$6n - 3\sum_{i=1}^{n} \nu_i + 2\sum_{i=1}^{n} \nu_i - 6 = 6 \tag{2.3}$$

or the familiar result

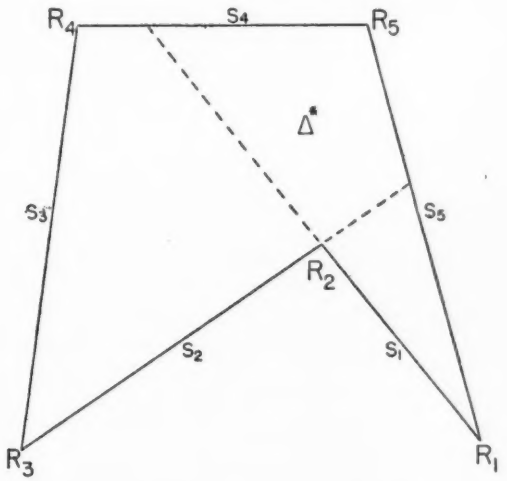
$$\sum_{i=1}^{n} (6 - \nu_i) = 12. \tag{2.4}$$

Except in the trivial case $n=3$, which implies that there are no interior vertices, each of the numbers ν_i exceeds two. Hence equation (2.4) implies that at least four of these numbers are less than six. Therefore

$$\nu_j < 6 \quad \text{for some } j \text{ of the set } \quad (4, \cdots, n). \tag{2.5}$$

Our proof of the above theorem involves an argument recurrent in n, the number of vertices. As a preliminary, we discuss a certain property of polygonal regions having fewer than six vertices.

3. Polygonal regions with $\nu<6$ vertices. Let $(R_1, \cdots, R_\nu)$ be the vertices of a polygonal region, Δ, where $\nu=3$, 4, or 5. Let $s_i=R_iR_{i+1}$ $(i=1, \cdots, \nu-1)$, and $s_\nu=R_\nu R_1$. If we regard Δ as the floor plan of a room with ν corners, then the result we need can be partly expressed by saying that there is always a non-vacuous locus, Δ^*, of vantage points from which one can observe all parts of the



room. Simple examples reveal that this statement is false for rooms with more than five corners. The locus Δ^*, in case $\nu<6$, will be referred to as the *core* of Δ.

LEMMA 3.1. *The core, Δ^*, of Δ is a non-vacuous, convex, polygonal region, and has a vertex in common with Δ.*

Proof. Let H_i $(i=1, \cdots, \nu)$ be that open half-plane with s_i on its boundary whose intersection with Δ also has s_i on its boundary. It is easy to show that Δ^* is the intersection

$$\Delta^* = H_1 \cdot H_2 \cdots H_\nu. \tag{3.1}$$

The reader can then complete the proof, breaking it into various cases classified according to (1) the value of ν and (2) the number and arrangement of re-entrant vertices of Δ.

(*A*) *We note that Δ^* is the locus of a point, P, such that the triangles PR_iR_{i+1} $(i=1, \cdots, \nu-1)$ and $PR_\nu R_1$ are all distinct and non-degenerate.*

In our application of this work, we will regard $(R_1, \cdots, R_\nu)$ as variable, subject to the requirement that they always determine a non-singular polygon, $R_1 \cdots R_\nu R_1$, bounding a region Δ. Both Δ and Δ^* vary continuously with the R's.

4. Proof of the theorem in the simplest case. Corresponding parts of isomorphic complexes will sometimes be denoted by the same symbol. Thus, for example, we will use Δ_j to denote the region covered by the star $S(A_j)$ in K_0, the region covered by $S(P_j)$ in K, or that covered by $S(B_j)$ in K_1. The context will make the meaning clear. The number j will be a fixed index such that condition (2.5) is fulfilled.

By a slight displacement of vertices, if necessary, let it be arranged that no three vertices of Δ_j be collinear. Let A_k be a vertex common to Δ_j and its core, Δ_j^* (Lemma 3.1). We will let the variable complex, K, have the initial position in which $P_i \equiv A_i$ $(i=1, \cdots, n)$. We first permit only P_j to vary, all other vertices of K being fixed. Then Δ_j^* is the locus of possible positions of P_j such that K remains isomorphic to K_0.

(A) Let P_j move along the line-segment from A_j to A_k, all vertices other than P_j remaining coincident with the corresponding A's. Until P_j coincides with P_k (at position A_k), K remains isomorphic to K_0. When coincidence is reached, there results a complex, K_0', with $(n-1)$ vertices. We will say that K_0 is reduced to K_0' by the coincidence of P_j and P_k. The structure of K_0' is completely determined by that of K and by the choice of the vertices P_j and P_k.

Hypothesis. For some value $n>3$, the deformation theorem of §2 holds with $(n-1)$ in place of n.

Verification of the hypothesis for $n=4$. If $n=4$, then the only complex with $(n-1)$ vertices is the fixed 2-cell $A_1A_2A_3$, plus boundary, and the deformation problem has a trivial solution.

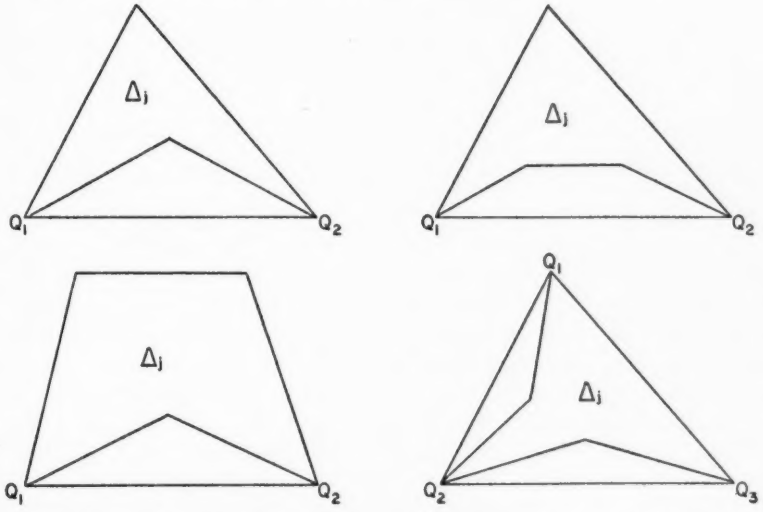
We will take the general step of the recurrency by deducing the existence of a deformation of K_0 into K_1 from the assumed existence of a deformation carrying K_0' [see (A) above] into an arbitrary isomorphic position subject to the restrictions of §2. The argument would be fairly straightforward if K_1 could necessarily be reduced to an isomorph of K_0' by moving P_j along the segment B_jB_k. However, K_1 may be such that B_k is not on the boundary of Δ_j^*. This difficulty will be obviated by the use of a complex, K_2, isomorphic to K_0 and K_1, in which Δ_j and Δ_j^* have all possible vertices in common.

CASE I. *No two non-adjacent vertices on the boundary, β_j, of Δ_j are joined by an edge of K_0.*

LEMMA 4.1. *In the present case, there exists a complex, K_2, isomorphic to K_0, with vertices $(C_1, \cdots, C_n)$, in which Δ_j is convex, with no three of its vertices collinear.*

Proof. We establish this lemma by commencing with the three vertices $C_i \equiv A_i$ $(i=1, 2, 3)$, and filling in the inner vertices in the following way. If all the vertices (A_1, A_2, A_3) are on β_j, which implies that $n=4$, then the conditions of the lemma are automatically fulfilled. If not, suppose $A_i \equiv C_i$ is not on β_j. Then the star $S(C_i)$, with vertices suitably denoted, can be introduced inside the triangle $A_1A_2A_3$, so that the part of the triangle not covered by $S(C_i)$ is a convex region, D, with no three vertices collinear. If all vertices of D correspond to

vertices of β_j, then C_j can be inserted anywhere in D, and the conditions of the lemma will be fulfilled. If not, then we fill in the missing cells of the star of some boundary vertex of D not on β_j; and we do it in such fashion that the portion of the triangle $A_1A_2A_3$ not covered by the sum of these cells and those previously introduced shall consist of a number of convex regions, with no three vertices collinear. We then deal with one of these regions exactly as we have just dealt with D. The recurrent process thus indicated leads ultimately to a situation where the entire triangle $A_1A_2A_3$ is covered, excepting a single convex region, Δ_j, whose vertices correspond to the vertices of Δ_j. We then introduce C_j into this region, and the conditions of the lemma are fulfilled.



(*B*) *Under the conditions of the foregoing lemma,* Δ_j *coincides with its core* Δ_j^*; *and* K_2 *can be reduced to a complex with* $(n-1)$ *vertices by a motion of* C_j *into any vertex on the boundary of* Δ_j.

Now suppose K_0 reducible to K_0' after the fashion described in (A) above. Then, by (B), K_2 is reducible to an isomorph, K_2', of K_0'. By the hypothesis of the recurrency, K_0' is deformable into K_2'. We now deduce that K_0 is deformable into K_2. Throughout any deformation of K_0' into K_2', the region Δ_j originally covered by $S(A_j)$ remains a polygon of $\nu_j<6$ sides. Instead of initially reducing K_0 to K_0', let P_j be moved to the centroid of the core, Δ_j^*, of Δ_j. Then let the vertices other than P_j move precisely as in some deformation of K_0' into K_2', while P_j remains at the centroid of the varying core Δ_j^*. Finally, let P_j move into C_j along a line-segment, while all other vertices remain fixed. This completes a deformation of K_0 into K_2. In similar fashion, K_1 can be deformed into K_2. It follows at once that K_0 is deformable into K_1, and the deformation theorem is proved for the present case.

5. Completion of the proof. CASE II. *Some non-adjacent vertices of the boundary, β_j, of Δ_j are joined by edges of K_0.* The various possibilities are shown in the preceding figures. Clearly, K_2 cannot be constructed, as in Case I, so that Δ_j will be convex. To say that Δ_j is convex is equivalent to saying that every vertex of Δ_j is also a vertex of Δ_j^*. In each of the present possibilities, those non-adjacent vertices [Q_i in the figures] of β_j which are joined by edges of K_0 cannot possibly be on the boundary of Δ_j^*. However, the following result holds and can be proved by a method similar to that used for Lemma 4.1.

LEMMA 5.1. *A complex, K_2, exists, isomorphic to K_0, in which Δ_j and Δ_j^* have in common all vertices save the non-adjacent vertices of β_j which are joined by edges of K_2.*

After Lemma 5.1 is proved, the discussion of Case II proceeds along the same lines as in Case I.

ASSOCIATED FREQUENCY DISTRIBUTIONS IN BIOMETRY*

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1. Introduction. The biometrician, studying minute structures, is frequently forced to limit his measurements to those which can be made on random plane sections of these structures. For example, sections of cells, alveoli, or glomeruli may be examined under the microscope and measured with respect to their diameters and areas. Such measurements yield some insight into the size of the original structures and are valuable for comparative purposes, but they fail to yield directly the statistical constants associated with a frequency distribution of the sizes of the structures themselves. There are instances† in the literature where an estimation of the mean volume of a set of structures has been made from the mean area of their random sections, but the present writer knows of no case in which an attempt has been made to determine additional statistical constants.

In view of its extensive applicability, the nature of the relationship existing between a distribution of the sizes of three dimensional structures and the distribution of the sizes of their plane sections merits more attention than it has received. The present discussion treats the problem as it applies to spheres and, further, suggests a method for adapting the results to nonspherical structures.

The mathematical framework required for the resolution of this problem is applicable to other similar problems. For example, the relationship between a distribution of velocity magnitudes and the distribution of their components in

* This problem was suggested by Dr. W. S. Hartroft, University of Western Ontario. A preliminary report was presented to the Royal Society of Canada, May 26, 1943, in conjunction with the report of Dr. W. S. Hartroft and Dr. C. C. Macklin on Pulmonic Alveolar Size.

† *Cf.*, for example, Roy Cohn, The postnatal growth of the lung, The Journal of Thoracic Surgery, vol. 9, 1939–40, pp. 274–277.

a given direction may be deduced by the same method that is developed here. That is to say, both problems lead to a consideration of the relationship existing between associated frequency distributions. It is desirable, therefore, to develop the relationship with respect to abstract variates. This will be done in sections 2 and 3, specific reference to the biometric problem being resumed in section 4. In that section we shall interpret the variate x, of sections 2 and 3, as the radius of a sphere and the variate y as the radius of a plane section of a sphere.

2. The general relationship. Let x be a variate which has a probability function, $\Phi(x)$, on the range (a, b). Suppose, further, that with each member of the universe of x there is associated a variate y which has a probability function, $\eta(x, y)$, on the range $(\alpha(x), \beta(x))$. If α is the greatest lower bound of $\alpha(x)$ and β the least upper bound of $\beta(x)$, we shall extend the range of definition of $\eta(x, y)$ to the rectangular region, $a \leqq x \leqq b$, $\alpha \leqq y \leqq \beta$, by stipulating that it be zero at every point not contained in the original region of definition.

Under simple sampling, then, the probability of a specified y associated with a specified x is given by $\eta(x, y)\Phi(x)$. The discussion may be made more general if we foresee the possibility of employing a method of sampling which is not simple but which weights the probability of x by a factor $s(x)$. When such sampling is performed, the probability function of the sample is $s(x)\Phi(x)$. Hence, the probability of a specified y associated with a specified x of the sample is $\eta(x, y)s(x)\Phi(x)$. If the probability of a specified y associated with any x of the sample is denoted by $\phi(y)$, we have

$$\phi(y) = \int_a^b \eta(x, y)s(x)\Phi(x)dx, \qquad \alpha \leqq y \leqq \beta. \tag{2.1}$$

If the functions $\phi(y)$, $\eta(x, y)$, and $s(x)$ are known, relation (2.1) is a Fredholm integral equation of the first kind for which a solution $\Phi(x)$ is required. Instead of employing the theory of integral equations, however, we shall approach the problem from the point of view of the practical statistician. That is, we shall suppose that a frequency distribution of y is known and shall attempt to determine therefrom the associated frequency distribution of x. Consequently, let us subdivide the interval (a, b) into n subintervals by the points $x_0=a, x_1, x_2, \cdots, x_n=b$, and the interval (α, β) into m subintervals by the points $y_0=\alpha, y_1, y_2, \cdots, y_m=\beta$.

The expected relative frequency of y on the subinterval $y_{i-1}<y\leqq y_i$ may be denoted by f_i and is given by

$$f_i = \int_{y_{i-1}}^{y_i} \phi(y)dy.$$

Whence, in view of (2.1),

$$f_i = \int_a^b \int_{y_{i-1}}^{y_i} \eta(x, y)s(x)\Phi(x)dydx, \qquad i = 1, 2, \cdots, m. \tag{2.2}$$

It is necessary, at this point, to make some assumption concerning the function $\Phi(x)$. Assuming, therefore, that $\Phi(x)$ is constant on (x_{j-1}, x_j), and denoting the relative frequency of x on that subinterval by F_j, we may write

$$\Phi(x) = \frac{F_j}{x_j - x_{j-1}}, \quad x_{j-1} < x \leqq x_j, \qquad j = 1, 2, \cdots, n.$$

Accordingly (2.2) may be written

$$\sum_{j=1}^{n} a_{i,j}F_j = f_i, \qquad i = 1, 2, \cdots, m, \tag{2.3}$$

where

$$a_{i,j} = \frac{1}{x_j - x_{j-1}} \int_{x_{j-1}}^{x_j} \int_{y_{i-1}}^{y_i} \eta(x, y)s(x)dydx.$$

The multiplication of the members of system (2.3) by the positive integer N will convert them from relations between expected *relative* frequencies to relations between expected frequencies from a sample of N. We may therefore interpret F_j and f_i as representing either frequencies or relative frequencies.

3. The inverse relationship. The relations (2.3) form a system of m equations in the n unknowns $F_1, F_2, \cdots, F_n$. We shall be interested in the case $m = n$, for which the existence of a unique solution depends on the value of the determinant $|a_{i,j}|$. It is frequently possible in specific cases to choose the subdivision points of the intervals (a, b) and (α, β) so that $|a_{i,j}|$ is triangular in form with the components of the main diagonal not zero. We shall assume, therefore, that

$$a_{i,j} = 0, \qquad i > j,$$
$$a_{j,j} \neq 0, \qquad j = 1, 2, \cdots, n,$$

and proceed to an explicit solution of system (2.3) which may now be written

$$\sum_{j=i}^{n} a_{i,j}F_j = f_i, \qquad i = 1, 2, \cdots, n. \tag{3.1}$$

If $i = n$ in (3.1) we get

$$F_n = \frac{1}{a_{n,n}} f_n.$$

Substituting this value for F_n in the other members of (3.1) and transposing, we obtain

$$\sum_{j=i}^{n-1} a_{i,j}F_j = f_{i,1}, \qquad i = 1, 2, \cdots, n-1, \tag{3.2}$$

where

$$f_{i,1} = f_i - \frac{a_{i,n}}{a_{n,n}} f_n.$$

Since (3.2) is of the same form as (3.1) this process evidently admits of repetition. Assume, then, that at the kth stage we have reduced the given set of equations to

$$\sum_{j=i}^{n-k} a_{i,j}F_j = f_{i,k}, \qquad i = 1, 2, \cdots, n-k, \tag{3.3}$$

where

$$f_{i,k} = f_{i,k-1} - \frac{a_{i,n-k+1}}{a_{n-k+1,n-k+1}} f_{n-k+1,k-1}. \tag{3.4}$$

If $i=n-k$, (3.3) yields

$$F_{n-k} = \frac{1}{a_{n-k,n-k}} f_{n-k,k}. \tag{3.5}$$

Substituting this value of F_{n-k} in (3.3) and transposing, we obtain

$$\sum_{j=i}^{n-k-1} a_{i,j}F_j = f_{i,k+1}, \qquad i = 1, 2, \cdots, n-k-1, \tag{3.6}$$

where

$$f_{i,k+1} = f_{i,k} - \frac{a_{i,n-k}}{a_{n-k,n-k}} f_{n-k,k}. \tag{3.7}$$

If $i=n-k-1$, (3.6) yields

$$F_{n-k-1} = \frac{1}{a_{n-k-1,n-k-1}} f_{n-k-1,k+1}. \tag{3.8}$$

Since (3.6), (3.7), and (3.8) may be obtained from (3.3), (3.4), and (3.5) respectively, by replacing k by $k+1$, the validity of the last mentioned set is established by induction when $k=0, 1, 2, \cdots, n-1$; when $k=0$ (3.4) must be replaced by $f_{i,0}=f_i$. The solution of the system (3.1) is given, therefore, by formulae (3.5) and (3.4) which, for convenience, may be rewritten in the equivalent form

$$F_{n-k} = \frac{1}{a_{n-k,n-k}} f_{n-k,k}, \qquad k = 0, 1, 2, \cdots, n-1, \tag{3.9}$$

$$f_{i,k} = f_{i,k-1} - R_{i,k-1} f_{n-k+1,k-1}, \qquad k = 1, 2, \cdots, n-1;\ i = 1, 2, \cdots, n-k, \tag{3.10}$$

where

$$R_{i,k-1} = \frac{a_{i,n-k+1}}{a_{n-k+1,n-k+1}}. \tag{3.11}$$

4. Application to plane sections of spheres. Consider a universe of spheres imbedded in a three dimensional space. Let their radii be represented by the variate x having a probability function $\Phi(x)$ on (a, b). Let a sample be selected by making random plane cuts through the imbedding space and considering each cut sphere a member of the sample. Such sampling is not simple since a sphere of radius x is x times as likely to be cut as a sphere of radius 1. We must therefore introduce a sampling function, $s(x)$, defined by

$$s(x) = cx. \tag{4.1}$$

The value of the constant c is evidently determined by the condition

$$\int_a^b cx\Phi(x)dx = 1.$$

However, since the relations (2.3) are linear, the effect of failing to choose c in this manner is to multiply each of the frequencies $F_1, F_2, \cdots, F_n$ by the same constant. This does not alter the essential character of the distribution, hence c may be chosen with a view to simplifying the computation.

The manner of selecting the sample of spheres has automatically yielded a random plane section from each sphere of the sample. Let the radii of these plane sections be represented by the variate y. The probability function of y from a sphere of radius x, denoted by $\eta(x, y)$, may be found as follows.

Let λ be the distance from the center of the sphere of a section of radius y. Then λ and y satisfy the relation

$$y^2 + \lambda^2 = x^2. \tag{4.2}$$

The probability function of λ is $1/x$, all distances from the center of the sphere being equally likely. Accordingly the probability differential, dp, is given by

$$dp = 1/x\, d\lambda.$$

Differentiating (4.2) and substituting for λ from the same relation yields the fact that

$$d\lambda = -\frac{ydy}{(x^2 - y^2)^{1/2}}. \tag{4.3}$$

Under this change of variate, therefore,

$$dp = -\frac{ydy}{x(x^2 - y^2)^{1/2}}.$$

The probability of finding y between y and $y+dy$ is the same as the probability of finding λ between λ and $\lambda+d\lambda$, where dy and $d\lambda$ are related by (4.3). Absorbing the negative sign in the differential dy, therefore, we have

$$\eta(x, y) = \begin{cases} \dfrac{y}{x(x^2 - y^2)^{1/2}} & \text{if } y < x \\ 0 & \text{if } y > x \end{cases} \tag{4.4}$$

Let $y_j = x_j$, $j = 1, 2, \cdots, n$, and denote by F_j and f_j the frequency of x and y, respectively, on the subinterval (x_{j-1}, x_j). The expected relationship between these frequencies is given by (2.3), where, in view of (4.1) and (4.4)

$$a_{i,j} = \begin{cases} \dfrac{c}{x_j - x_{j-1}} \displaystyle\int_{x_{j-1}}^{x_j} \int_{x_{i-1}}^{x_i} (x^2 - y^2)^{-1/2} y\,dy\,dx, & \text{if } i < j \\ \dfrac{c}{x_j - x_{j-1}} \displaystyle\int_{x_{j-1}}^{x_j} \int_{x_{j-1}}^{x} (x^2 - y^2)^{-1/2} y\,dy\,dx, & \text{if } i = j \\ 0, & \text{if } i > j. \end{cases}$$

Integrating with respect to y, and defining $I_{i,j}$ by the relation

$$I_{i,j} = \begin{cases} \dfrac{1}{x_j - x_{j-1}} \displaystyle\int_{x_{j-1}}^{x_j} (x^2 - x_i^2)^{1/2} dx, & \text{if } i < j \\ 0, & \text{if } i \geqq j, \end{cases}$$

it may be verified that

$$a_{i,j} = c(I_{i-1,j} - I_{i,j}), \qquad i, j = 1, 2, \cdots, n.$$

It follows that $|a_{i,j}|$ is triangular in form with the elements of the main diagonal not zero. The hypotheses of section 3 are therefore satisfied and the inverse relationship is given by (3.9), (3.10), and (3.11).

Although the formula for the evaluation of $I_{i,j}$ is familiar, the task of evaluating $(n-1)^2/2$ such quantities is sufficiently arduous to discourage the application of these results to a given collection of data. For this reason it is desirable to mold the results so that they admit of readier application, provided this can be done without extensive sacrifice of accuracy. To this end, we shall abandon the normal procedure of grouping the sections in radius classes of equal width in favor of area classes of equal width. While this will evidently have the effect of grouping the resulting spheres in radius classes of unequal width, the inconvenience so occasioned is not great. Let the subdivision points, therefore, be defined by the relation

$$x_j^2 = jw, \qquad j = 0, 1, \cdots, n, \tag{4.5}$$

where w is defined by

$$b^2 = nw.$$

An approximate evaluation of $I_{i,j}$ may be obtained by using the trapezoidal rule‡ for approximate integration. Thus, if the interval (x_{j-1}, x_j) is divided into

‡ *Cf.*, for example, Smith, Salkover, and Justice, Calculus, p. 322.

m equal parts the rth subdivision point is, by virtue of (4.5),

$$w^{1/2}\left[(j-1)^{1/2}+\frac{r}{m}\left\{j^{1/2}-(j-1)^{1/2}\right\}\right], \qquad r=0, 1, \cdots, m.$$

The square of this quantity, evaluated for substitution purposes, is

$$\frac{w}{m^2}\left[(m-r)^2(j-1)+2(m-r)r\left\{j(j-1)\right\}^{1/2}+r^2j\right].$$

If, in the latter expression, $\left\{j(j-1)\right\}^{1/2}$ is replaced by $j-\frac{1}{2}$ it reduces to

$$\frac{w}{m}\left[m(j-1)+r\right].$$

The error so introduced is negligible if $j\geqq 5$ and the practical approach of actually calculating the integrals associated with j less than 5 will show that the replacement is justified in these cases also. The approximate functional value of $(x_2^2-x_1^2)^{1/2}$ at the rth subdivision point is therefore seen to be

$$\frac{w^{1/2}}{m^{1/2}}\left[m(j-i-1)+r\right]^{1/2},$$

and the application of the trapezoidal rule yields

$$I_{i,j}\doteq\frac{w^{1/2}}{2m^{3/2}}\left[\left\{m(j-i-1)\right\}^{1/2}+2\sum_{r=1}^{m-1}\left\{m(j-i-1)+r\right\}^{1/2}+\left\{m(j-i)\right\}^{1/2}\right], \text{ if } i<j.$$

Thus the approximate evaluation of $I_{i,j}$ is, in effect, accomplished by simply adding square roots of successive integers. The most significant feature of this approximation is that it appears as a function of $j-i$. This implies that $a_{i,j}$ is a function of $j-i$, that $a_{i,n-k+1}$ is a function of $k+i$, and that $a_{n-k,n-k}$ is a constant. In view of the last remark we may choose c so that $a_{n-k,n-k}=1$, whereupon formula (3.9) reduces to

$$F_{n-k}=f_{n-k,k}. \tag{4.6}$$

Again it may be verified, if we recall formula§ (3.11), that $R_{i,k-1}$ is a function of $k+i$ alone for any fixed n. The significance of this fact becomes evident if we envisage the actual steps in the reduction of a given distribution of sectional radii. The *reduction factors* to be applied to the original frequencies f_{n-1}, f_{n-2}, $\cdots$, f_1 are, respectively, $R_{n-1,0}$, $R_{n-2,0}$, $\cdots$, $R_{1,0}$, and yield, through (3.10), the *reduced frequencies* $f_{n-1,1}, f_{n-2,1}, \cdots, f_{1,1}$. To these latter, with the first member deleted, we apply the reduction factors $R_{n-2,1}, \cdots, R_{1,1}$, which are respectively equal to the members of the first set with the last one, $R_{1,0}$, deleted. Similarly, if the last two members of the first set are deleted, we obtain the third set

§ Formula (3.11) reduces to $R_{i,k-1}=a_{i,n-k+1}$. However, since $R_{i,k-1}$ is readily seen to be independent of c, this simplification does not serve to reduce the labor of computation.

of reduction factors, *etc.* Thus the determination of $n-1$ reduction factors, requiring the calculation of n quantities of the form $I_{i,j}$, yields the complete set of reduction factors.

The use of successive values of m in the trapezoidal formula will reveal that if $m=3$ a satisfactory approximation to $I_{i,j}$ is obtained in every case except when $i=j-1$. In this instance it will be found desirable to use 16 for m. The first 15 reduction factors evaluated in this way will be found to be

$R_{n-k-1,k} = .835$	$R_{n-k-6,k} = .308$	$R_{n-k-11,k} = .227$
$R_{n-k-2,k} = .543$	$R_{n-k-7,k} = .285$	$R_{n-k-12,k} = .218$
$R_{n-k-3,k} = .438$	$R_{n-k-8,k} = .267$	$R_{n-k-13,k} = .209$
$R_{n-k-4,k} = .378$	$R_{n-k-9,k} = .251$	$R_{n-k-14,k} = .201$
$R_{n-k-5,k} = .338$	$R_{n-k-10,k} = .238$	$R_{n-k-15,k} = .195$

The retention of three figures in these values may be justified by employing exact integration to evaluate a few selected factors. The set here given is sufficient for a distribution of no more than sixteen classes; if distributions with more classes are to be handled additional factors may be easily and quickly calculated.

The actual reduction of section radius frequencies to the corresponding sphere radius frequencies may be accomplished in tabular form as illustrated by the following hypothetical example.

DISTRIBUTION OF RADII OF MINUTE SPHERICAL STRUCTURES DERIVED FROM A DISTRIBUTION OF AREAS OF THEIR RANDOM SECTIONS
(all figures rounded off to the nearest integer)

Section Area Classes (1000 sq. microns)	Equivalent Radius Classes (microns)	Mid-point	Section Frequency f_i	$R_{i,0}f_0$	$f_i - R_{i,0}f_0 = f_{i,1}$	$R_{i,1}f_{i,1}$	$f_{i,1} - R_{i,1}f_{i,1} = f_{i,2}$	$R_{i,2}f_{i,2}$	$f_{i,2} - R_{i,2}f_{i,2} = f_{i,3}$	$R_{i,3}f_{i,3}$	$f_{i,3} - R_{i,3}f_{i,3} = f_{i,4}$	Sphere Frequency $F_i = f_{i,n-i}$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)
40–50	113–126	120	*8*									8
30–40	98–113	106	38	7	*31*							31
20–30	80– 98	89	109	4	105	26	*79*					79
10–20	56– 80	68	124	4	120	17	103	66	*37*			37
0–10	0– 56	28	93	3	90	14	76	43	33	31	*2*	2

Since the distribution contains five classes, only the first four reduction factors are used, namely, .835, .543, .438, and .378. These, multiplied in order by the top member of column (4), constitute column (5). Column (6) is obtained by subtracting the members of column (5) from the corresponding members of column (4). Column (7), like column (5), is formed by multiplying the first three reduction factors in order by the top member of column (6). Column (8), like column (6), is the result of subtracting the members of column (7) from the corresponding members of column (6). Continuing in this way the triangular array is completed by column (12). Recalling formula (4.6), the sphere frequencies of column (13) are seen to be the top members of columns (4), (6), (8), (10), and (12).

The calculation of such statistical constants as the mean and the standard deviation must be performed from first principles, since the various short methods that have been devised assume class intervals of equal width.

5. Nonspherical structures. The practical statistician will seldom be presented with data derived from sections of spherical structures. The cells and other parts of plants and animals, for example, are not in general spherical. Nevertheless, many such structures may be regarded as approximations to spheres without significant sacrifice of accuracy. When this is the case, the applicability of the above results is immediate. Furthermore, it may be argued qualitatively that there is justification for applying them even when the structures are known to differ markedly from spheres. In support of this it may be pointed out that the average area of the sections of an ellipsoid of revolution, whose axes are in the ratio 3:2, is about 8% less than the average area of the sections of a sphere of equal volume. This is equivalent to a deviation of about 4% when reduced to linear measure.

When dealing with nonspherical structures the areas of the sections must be determined‖ and arranged in classes of equal width. Corresponding to each class boundary the radius of a circle of equivalent area may be determined. These values will then serve as class boundaries for the radii of the spheres. The spheres should not be regarded as approximations in form to the original structures but rather as being equivalent to them in volume.

The derivation of analogous results for geometric figures other than the sphere is clearly possible. However in the absence of regular structures it would seem logical to apply the procedure outlined above and to make a critical appraisal of the conclusions.

6. Conclusion. The relationship between associated variates, derived in sections 3 and 4, is applicable to a variety of problems. The case of velocity magnitudes and their components has already been mentioned and will be developed in a subsequent paper. Another application is the derivation of a distribution of

‖ Areas may be found by using a planimeter on camera lucida drawings, or they may be estimated from linear measurements made on the sections.

family sizes from data which is limited to the family order of birth of a group of children. The treatment required by each problem owes its individuality only to the fact that the method of application may be designed to simplify the computation. Such simplification is attained by a suitable choice of class boundaries for the two variates.

ON THE DENSEST PACKING OF CIRCLES

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1. Introduction. We show in this note that at most $A/\sqrt{12}$ circles of radius 1 can be placed in a convex polygon of area A and with angles not greater than $\frac{2}{3}\pi$,* such that no two of the circles overlap. This upper bound for the number of inscribed circles is the best possible one, but the restriction on the angles is essential. Somewhat similar results have been obtained, using an entirely different method, by A. Thue.† More recently, the problem was studied by L. Fejes‡ and R. Rado.§

Apart from a simple application of differential calculus, our method is elementary. The proof is based on two lemmas which have a certain interest in themselves.

2. The convex polygon $S(P)$. Let Σ be a set of points P, P_1, P_2, $\cdots$ in the plane π, such that the distance $\overline{P'P''}$ of any two different points P', P'' of Σ is at least 2. Hence, if $C(P)$, $C(P_1)$, $C(P_2)$, $\cdots$ are the circles of radius 1 and centers P, P_1, P_2, $\cdots$, then no two of these circles overlap.

Let K be any circle in Π, say of radius r, and let K' be the concentric circle of radius $r+1$. If now P', P'', $\cdots$, $P^{(l)}$ are any points of Σ in K, then the circles $C(P')$, $C(P'')$, $\cdots$, $C(P^{(l)})$ are contained in K'; their total area $l\pi$ is therefore not larger than the area $(r+1)^2\pi$ of K', and so $l \leqq (r+1)^2$. Hence *every circle contains at most a finite number of points of* Σ.

For every point P of Σ, denote by $S(P)$ the set of all points Q in Π for which

$$\overline{PQ} \leqq \overline{P_kQ} \qquad (k = 1, 2, \cdots). \tag{1}$$

The circle $C(P)$ is a subset of $S(P)$. For, if Q lies in $C(P)$, then

$$\overline{P_kQ} \geqq \overline{PP_k} - \overline{PQ} \geqq 2 - 1 = 1 \geqq \overline{PQ}, \quad \text{since} \quad \overline{PQ} \leqq 1, \ \overline{PP_k} \geqq 2.$$

Denote by Λ_k the locus of all points Q for which $\overline{PQ} = \overline{P_kQ}$. Evidently Λ_k is the line perpendicular to the line PP_k which intersects the segment PP_k at

* All angles in this paper are measured in radians. Moreover, we use the same symbols for the angles and their measures in radians.

† See his note read at the Scandinavian Mathematical Congress of 1892, and the paper, Norske Vid. Selsk. Skr. 1910, No. 1.

‡ Math. Z. 46, 1940, 83–85.

§ Dr. Rado was so kind as to give us the reference to Fejes's note, and he also informed us of his, as yet unpublished, results.

its center, say $P^{(k)}$. The points Q satisfying $\overline{PQ} \leqq \overline{P_kQ}$ form the semi-plane determined by Λ_k containing P, and this is a convex region. Therefore, from (1), $S(P)$ *is a closed convex region.* (Fig. 1.)

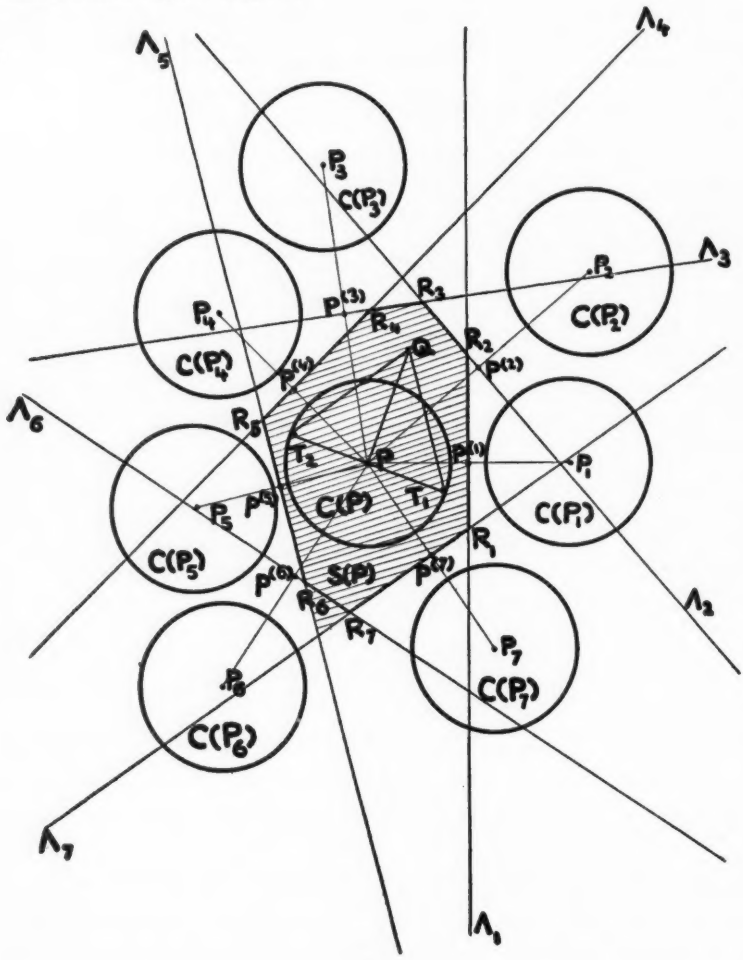

FIG. 1

We assume from now on that $S(P)$ is of *finite area*, $A(P)$ say. Let $Q \neq P$ be a point in $S(P)$, and denote by T_1 and T_2 the two endpoints of the diameter of $C(P)$ perpendicular to PQ. Since $S(P)$ is convex, and since the three points Q, T_1, T_2 belong to $S(P)$, the whole triangle T_1QT_2 is a subset of $S(P)$, and so its area is not greater than $A(P)$. Now the area of this triangle is

$$\tfrac{1}{2}\overline{T_1T_2} \times \overline{PQ} = \overline{PQ}, \quad \text{since} \quad \overline{T_1T_2} = 2,$$

whence

$$\overline{PQ} \leq A(P) \quad \textit{for every point } Q \textit{ of } S(P). \tag{2}$$

By (2), $S(P)$ lies in the circle $C'(P)$ of center P and radius $A(P)$. Hence it has points in common only with those lines Λ_k which touch or pass through $C'(P)$. As $P^{(k)}$ is the point of Λ_k nearest to P, this requires that

$$\overline{PP^{(k)}} \leq A(P), \quad \textit{i.e.,} \quad \text{that} \quad \overline{PP}_k = 2\overline{PP^{(k)}} \leqq 2A(P).$$

But since only a finite number of points P_k of Σ satisfy this inequality, the boundary of $S(P)$ meets only a finite number of the lines Λ_k. Therefore $S(P)$ *is a convex polygon*, say with the n sides $\Lambda_1, \Lambda_2, \cdots, \Lambda_n$, and the n vertices $R_1, R_2 \cdots, R_n$. We choose the notation such that these sides and vertices lie on the boundary of $S(P)$ in the order of their indices, and that R_k and R_{k+1} are the vertices on Λ_k, and so Λ_{k-1} and Λ_k are the sides through R_k. (The indices 0 and $n+1$ must be replaced by n and 1, respectively.)

3. A fundamental lemma. The lines from P to the n vertices $R_1, R_2, \cdots, R_n$ split $S(P)$ into the n triangles

$$R_1PR_2, \ R_2PR_3, \cdots, \ R_nPR_1,$$

say of areas

$$a_1, \ a_2, \cdots, \ a_n.$$

Let further the angles at P of these triangles be

$$\alpha_1, \ \alpha_2, \cdots, \ \alpha_n,$$

respectively. Then, evidently,

$$a_1 + a_2 + \cdots + a_n = A(P), \tag{3}$$

and

$$\alpha_1 + \alpha_2 + \cdots + \alpha_n = 2\pi. \tag{4}$$

In the next paragraphs, we prove the following

LEMMA 1: *For every index k,*

$$a_k \geqq \frac{\sqrt{3}}{\pi}\alpha_k, \tag{5}$$

with equality if and only if the two circles $C(P)$ and $C(P_k)$ touch each other, and are both touched by the circles $C(P_{k-1})$ and $C(P_{k+1})$.

For the proof of (5), put (Fig. 2)

$$x = \overline{PP^{(k)}} = \tfrac{1}{2}\overline{PP}_k, \quad \text{so that} \quad x \geqq 1.$$

Further denote by Γ the circle of center P and radius x; then Λ_k is a tangent of

Γ. The two lines from P to R_k and R_{k+1} cut off Γ a sector of angle α_k, hence of area $\frac{1}{2}x^2\alpha_k$. Since this sector lies entirely in the triangle R_kPR_{k+1} of area a_k, the

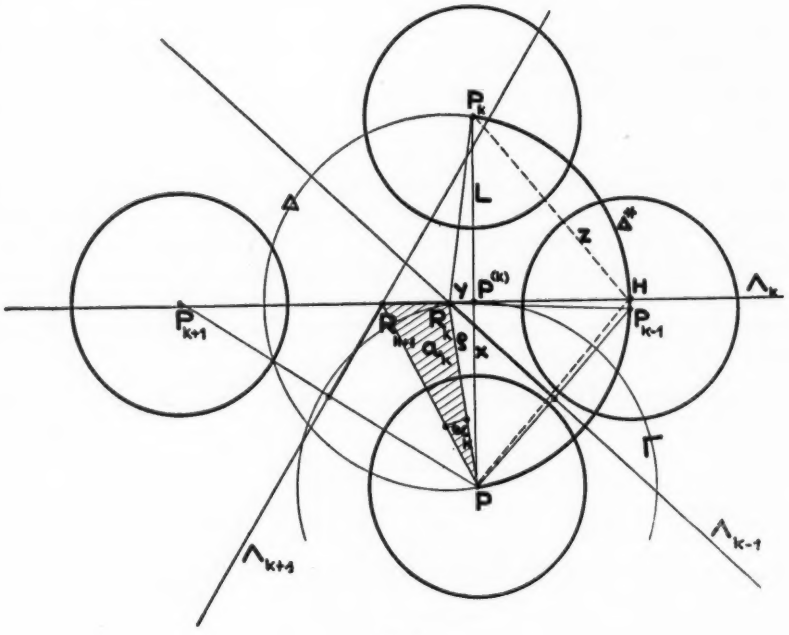

FIG. 2

inequality $a_k > \frac{1}{2}x^2\alpha_k$ holds. Hence if $\frac{1}{2}x^2 \geqq \sqrt{3}/\pi$, *i.e.*, if $x \geqq \sqrt[4]{12/\pi^2} > 1$, then (5) is satisfied with the sign "$>$" instead of "$\geqq$," and so the assertion is proved, since $C(P)$ and $C(P_k)$ obviously do not touch each other.

We may therefore exclude this case, and assume from now on that

$$1 \leqq x < \sqrt[4]{\frac{12}{\pi^2}} < \sqrt{\frac{4}{3}}. \tag{6}$$

We now prove the result:

The point $P^{(k)}$ is interior to the segment R_kR_{k+1}.

If this is not so, then R_k, R_{k+1} are two distinct points of the line Λ_k, lying on the same side of $P^{(k)}$. Hence one of these points, say R_k, is nearer than the other one to $P^{(k)}$; and R_k possibly coincides with $P^{(k)}$.

By our notation, Λ_{k-1} and Λ_k meet at R_k. Since all points of Λ_{k-1} are equidistant from P and P_{k-1}, this implies that $\overline{PR}_k = \overline{P_{k-1}R}_k$, and so P and P_{k-1} lie on the circle Δ of center R_k and radius $\rho = \overline{PR}_k$. This circle contains also P_k, and is divided into two arcs by the line L joining P and P_k. Let Δ^* be the arc which

meets Λ_k at a point, H say, separated from R_{k+1} by R_k; then Δ^* is separated from R_{k+1} by L. Since the only common point of the line Λ_{k-1} and the angle PR_kR_{k+1} is the vertex R_k of this angle, it follows that the image P_{k-1} of P in Λ_{k-1} is separated from R_{k+1} by both the line PR_k and its image R_kP_k in $\Lambda_k = R_kR_{k+1}$; hence P_{k-1} lies on Δ^*. On putting

$$z = \overline{PH} = \overline{P_kH},$$

it is evident that for every point Q on Δ^*,

$$\min (\overline{PQ}, \overline{P_kQ}) \leqq z,$$

and so, in particular,

$$\min (\overline{PP}_{k-1}, \overline{P_kP}_{k-1}) \leqq z.$$

Therefore, by the definition of Σ,

$$z \geqq 2. \tag{7}$$

Next put

$$\overline{R_kP^{(k)}} = y,$$

so that

$$\rho^2 = x^2 + y^2 \quad \text{and} \quad \overline{P^{(k)}H} = \overline{R_kH} - \overline{R_kP^{(k)}} = \rho - y = \sqrt{x^2 + y^2} - y \leqq x,$$

since $\sqrt{x^2+y^2} \leqq x+y$. Hence

$$z^2 = \overline{PP^{(k)}}^2 + \overline{P^{(k)}H}^2 \leqq x^2 + x^2 = 2x^2,$$

whence, from (6),

$$z \leqq \sqrt{2}\, x < \sqrt{2}\sqrt{\tfrac{4}{3}} < 2,$$

contrary to (7). This contradiction proves the result.

From the above, the line $PP^{(k)}$ divides the triangle R_kPR_{k+1} into the two triangles $R_kPP^{(k)}$, say of area b_k, and $P^{(k)}PR_{k+1}$, say of area c_k, so that

$$b_k + c_k = a_k. \tag{8}$$

The line $PP^{(k)}$ splits α_k into two angles $\beta_k = R_kPP^{(k)}$ and $\gamma_k = P^{(k)}PR_{k+1}$ satisfying

$$\beta_k + \gamma_k = \alpha_k. \tag{9}$$

By (8) and (9), Lemma 1 is proved if we can show that

$$b_k \geqq \frac{\sqrt{3}}{\pi}\beta_k, \tag{10}$$

and

$$c_k \geqq \frac{\sqrt{3}}{\pi}\gamma_k, \tag{11}$$

with equality if and only if $C(P)$ and $C(P_k)$ touch each other, and are touched

by $C(P_{k-1})$ in the case of the inequality (10), and by $C(P_{k+1})$ in inequality (11). It suffices to prove the assertion (10), since (11) can be treated likewise.

As before, we denote by Δ the circle of center R_k and radius $\rho = \overline{PR_k}$ (Fig. 3); then P, P_{k-1} and P_k lie on this circle. The line L joining P and P_k divides Δ into two unequal arcs (since $P^{(k)}$ and R_k are distinct); let Δ^* be the larger one, *i.e.*

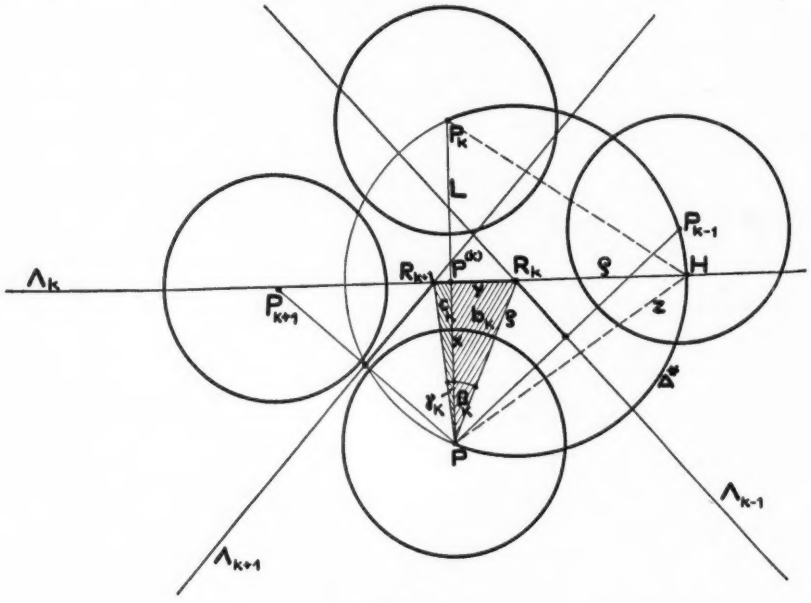

FIG. 3

that arc which is separated from R_{k+1} by L. Then, by an argument already used we see that P_{k-1} lies on Δ^*.

Denote again by H the point where Δ^* and Λ_k intersect, and put

$$y = \overline{R_k P^{(k)}}, \qquad z = \overline{PH} = \overline{P_k H}.$$

Then $\rho = \sqrt{x^2+y^2}$, and we find as in §7 that

$$z \geqq 2, \tag{12}$$

but now

$$\begin{aligned} \overline{P^{(k)}H} &= \overline{R_k H} + \overline{P^{(k)}R_k} = \rho + y = \sqrt{x^2+y^2} + y, \\ z^2 = \overline{PH}^2 &= \overline{PP^{(k)}}^2 + \overline{P^{(k)}H}^2 = x^2 + (\sqrt{x^2+y^2}+y)^2 \\ &= 2(x^2 + y^2 + y\sqrt{x^2+y^2}). \end{aligned} \tag{13}$$

From (12) and (13),

$$x^2 + y^2 + y\sqrt{x^2+y^2} \geqq 2,$$

whence, on solving for y,

$$y \geqq \frac{2 - x^2}{\sqrt{4 - x^2}}.$$

Since further

$$b_k = \tfrac{1}{2}xy, \qquad \beta_k = \sin^{-1} \frac{y}{\sqrt{x^2 + y^2}},$$

the inequality (10) can be written as

$$F(x, y) \geqq 0, \quad \text{where} \quad F(x, y) = \pi xy - \sqrt{12} \sin^{-1} \frac{y}{\sqrt{x^2 + y^2}}.$$

Hence the assertion (10) is proved if we can show that

$$F(x, y) > 0, \tag{14}$$

if

$$1 \leqq x < \sqrt{\tfrac{4}{3}} \quad \text{and} \quad y \geqq \frac{2 - x^2}{\sqrt{4 - x^2}}, \quad \text{and either} \quad x > 1 \quad \text{or} \quad y > \frac{2 - x^2}{\sqrt{4 - x^2}}, \quad \text{or both.} \tag{15}$$

For in the excluded case, we have

$$x = 1, \qquad y = \frac{2 - x^2}{\sqrt{4 - x^2}} = \sqrt{\tfrac{1}{3}}, \qquad z = 2, \qquad F(x, y) = 0,$$

and so the equality sign holds in (10). Moreover, since $z = 2$, every point $Q \neq H$ on Δ^* has a distance less than 2 from either P or P_k; therefore P_{k-1} must coincide with H, *i.e.*, the three circles $C(P)$, $C(P_{k-1})$, $C(P_k)$ touch in pairs.

When (15) is satisfied, then

$$\frac{x^2 + y^2}{x} \frac{\partial F(x, y)}{\partial y} = \pi(x^2 + y^2) - \sqrt{12} \geqq \pi\left(x^2 + \frac{(2 - x^2)^2}{4 - x^2}\right) - \sqrt{12}$$

$$= \frac{4\pi}{4 - x^2} - \sqrt{12} \geqq \frac{4\pi}{4 - 1} - \sqrt{12} > 0.$$

Hence $F(x, y)$ is a *strictly increasing* function of y. To prove (14), it therefore suffices to show that the function

$$f(x) = F\left(x, \frac{2 - x^2}{\sqrt{4 - x^2}}\right) = \pi \frac{x(2 - x^2)}{\sqrt{4 - x^2}} - \sqrt{12} \sin^{-1} (1 - \tfrac{1}{2}x^2)$$

is *strictly increasing*, since $f(1) = 0$. Now

$$f(x) = 2g(t), \quad \text{where} \quad t = 1 - \tfrac{1}{2}x^2, \qquad g(t) = \pi t \sqrt{\frac{1 - t}{1 + t}} - \sqrt{3} \sin^{-1} t,$$

and where, by (15), t satisfies the inequality

$$\tfrac{1}{3} = 1 - \tfrac{2}{3} < t < 1 - \tfrac{1}{2} = \tfrac{1}{2}.$$

On differentiating, we get

$$\frac{df(x)}{dx} = -2x\frac{dg(t)}{dt}, \qquad \frac{dg(t)}{dt} = \frac{1-t}{(1-t^2)^{3/2}}\left[\pi - (\pi t + \sqrt{3})(1+t)\right].$$

Now $3<\pi<10/3$ and $\sqrt{3}>3/2$, hence

$$(\pi t + \sqrt{3})(1+t) \geqq (3\cdot\tfrac{1}{3} + \tfrac{3}{2})(1+\tfrac{1}{3}) = \tfrac{10}{3} > \pi,$$

and therefore

$$\frac{dg(t)}{dt} < 0, \qquad \frac{df(x)}{dx} > 0,$$

as asserted. This concludes the proof of Lemma 1.

4. A second lemma. We now prove the following

LEMMA 2. *The convex polygon $S(P)$, defined in §2, is of area*

$$A(P) \geqq \sqrt{12}, \tag{16}$$

with equality if and only if $C(P)$ is touched by six circles $C(P_1), \cdots, C(P_6)$, whose centers form a regular hexagon of side 2 and center P.

From (3), (4), and Lemma 1, the inequality (16) follows at once. We see moreover, that the equality sign can hold only if

$$a_k = \frac{\sqrt{3}}{\pi}\alpha_k \qquad (k = 1, 2, \cdots, n),$$

i.e., if each circle $C(P_k)$ in the set

$$C(P_1), C(P_2), \cdots, C(P_n)$$

is touched by both $C(P_{k-1})$ and $C(P_{k+1})$, and itself touches $C(P)$. Hence

$$\overline{PP_1} = \overline{PP_2} = \cdots = \overline{PP_n} = \overline{P_1P_2} = \overline{P_2P_3} = \cdots = \overline{P_nP_1} = 2.$$

Now the regular hexagon, and no other regular polygon, has the property that its side is equal to the radius of the circumscribed circle; therefore n must be equal to 6, and the circles $C(P), C(P_1), \cdots, C(P_6)$ must be situated as asserted.

5. The theorem. We can now prove the following theorem.

Let Π_0 be a convex polygon of angles not greater than $\frac{2}{3}\pi$, hence of at most six sides. If A denotes the area of Π_0, then at most $A/\sqrt{12}$ circles of radius 1 can be placed in Π_0 such that no two of these circles overlap.

Proof (Fig. 4): Let $C_{01}, C_{02}, \cdots, C_{0l}$ be the circles placed in Π_0, and let $\Lambda_1, \Lambda_2, \cdots, \Lambda_n$ be the sides of Π_0. Denote further by Π_h the polygon symmetri-

cal to Π_0 in Λ_h, and by $C_{h1}, C_{h2}, \cdots, C_{hl}$ the circles symmetrical to $C_{01}, C_{02}, \cdots, C_{0l}$ in Λ_h; these new circles lie in Π_h. No two of the $m=(n+1)l$ circles

$$C_{01}, C_{02}, \cdots, C_{0l}, \cdots, C_{n1}, C_{n2}, \cdots, C_{nl} \tag{17}$$

overlap. This is obvious, from the hypothesis, for any two circles in the *same* polygon Π_h, and also for any two circles one of which lies in Π_0. To prove the assertion for two circles lying in two different polygons $\Pi_1, \Pi_2, \cdots, \Pi_n$, it is obviously sufficient to show that no two of these polygons, Π_h and Π_k say $(h \neq k)$, overlap. This is evident if Λ_h, Λ_k are parallel, since then Π_h, Π_k lie on opposite

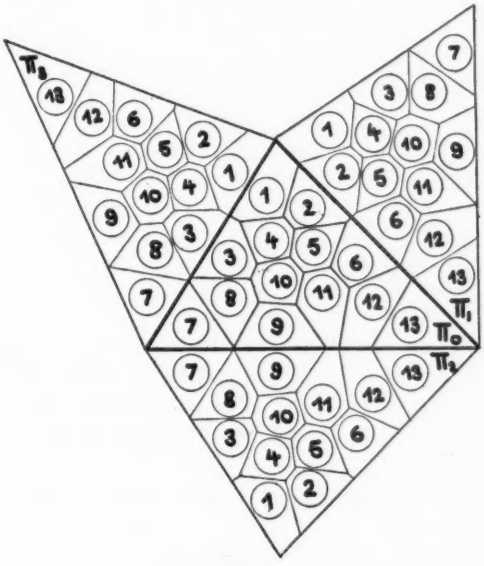

FIG. 4

sides of the strip determined by these lines. If Λ_h, Λ_k meet at a point, O say, we consider that angle ϕ of vertex O formed by these two lines which contains Π_0. Then $\phi \leqq \frac{2}{3}\pi$, from the hypothesis for Π_0, and so the images of ϕ in Λ_h and Λ_k are two non-overlapping angles of vertex O. Since one of these angles contains Π_h and the other contains Π_k, it follows that Π_h and Π_k do not overlap.

Denote now by Σ the set of the centers

$$P_{01}, P_{02}, \cdots, P_{0l}, \cdots, P_{n1}, P_{n2}, \cdots, P_{nl}$$

of the circles (17), so that Σ has the properties stated in §1. In accordance with the definition in §1, we form the convex regions

$$S(P_{01}), \cdots, S(P_{0l}), \cdots, S(P_{n1}), \cdots, S(P_{nl});$$

these m regions together fill out the plane, and no two of them overlap.

We show next that all l polygons

$$S(P_{01}), S(P_{02}), \cdots, S(P_{0l}) \tag{18}$$

lie in Π_0. For consider, *e.g.* $S(P_{01})$, and take any side Λ_h of Π_0. If P_{h1} is the point symmetrical to P_{01} in Λ_h, then every point Q of $S(P_{01})$ is at least as near to P_{01} as to P_{h1}, hence lies on the same side of Λ_h as P_{01}. This is true for all indices $h=1, 2, \cdots, n$, and so Q lies in Π_0. We deduce then that the total area of the l polygons (18) is just equal to the area A of Π_0.

By Lemma 2, each polygon $S(P_{0i})$ is at least of area $\sqrt{12}$. This implies that

$$l\sqrt{12} \leqq A, \qquad l \leqq \frac{A}{\sqrt{12}},$$

as asserted. Moreover, our proof shows that *the equality*

$$l = \frac{A}{\sqrt{12}}$$

holds, if and only if Π_0 *is the sum set of a finite number of regular hexagons of side* 2.

We finally remark that the Theorem is untrue for polygons with angles greater than $\frac{2}{3}\pi$. For instance, the regular heptagon circumscribed to a circle of radius 1 is of area less than $\sqrt{12}$.

AN APPLICATION OF A THEOREM OF SYLVESTER

L. R. WILCOX, Illinois Institute of Technology

In this paper a little used theorem on determinants due to Sylvester is employed to prove a theorem in the geometry of curves which generalizes the well known result that the tangent plane to a developable surface is the same at all points of a fixed generator.*

1. The theorem of Sylvester. Let a matrix $(a_{ij}; i, j=0, \cdots, m)$ be given, and define

$$A_h \equiv \begin{vmatrix} a_{00} & \cdots & a_{0h} \\ \cdots & \cdots & \cdots \\ a_{h0} & \cdots & a_{hh} \end{vmatrix} \qquad (h = 0, \cdots, m),\ A \equiv A_m,$$

$$b_{rs}^{(h)} \equiv \begin{vmatrix} a_{00} & \cdots & a_{0h} & a_{0s} \\ \cdots & \cdots & \cdots & \cdots \\ a_{h0} & \cdots & a_{hh} & a_{hs} \\ a_{r0} & \cdots & a_{rh} & a_{rs} \end{vmatrix} \qquad (r, s = h+1, \cdots, m),$$

$$B^{(h)} \equiv \left| b_{rs}^{(h)} \right| \qquad (h = 0, \cdots, m-1).$$

* E. P. Lane, Projective Differential Geometry of Curves and Surfaces, Chicago, 1932, pp. 37–38.

Sylvester's theorem states that for $0 \leqq h \leqq m-1$

$$A_h^{m-h-1} \cdot A = B^{(h)}. \tag{1}$$

In particular, if $h = m-2$, we have

$$\begin{vmatrix} a_{00} & \cdots & a_{0,m-2} \\ \cdots & \cdots & \cdots \\ a_{m-2,0} & \cdots & a_{m-2,m-2} \end{vmatrix} \cdot A = \begin{vmatrix} b_{m-1,m-1}^{(m-2)} & b_{m-1,m}^{(m-2)} \\ b_{m,m-1}^{(m-2)} & b_{mm}^{(m-2)} \end{vmatrix}. \tag{2}$$

It is this case with which we shall be concerned. The proof will be omitted; it is made inductively with the help of Laplace's expansion.†

2. The theorem on developable surfaces. Let us consider briefly a simple problem in three-dimensional projective geometry. Let a curve C be generated by a point $P(t)$ whose homogeneous coordinates are

$$x_i = f_i(t) \qquad (i = 0, \cdots, 3).$$

The plane $x_0 = 0$ and the line $x_0 = x_1 = 0$ are called S_2 and S_1 respectively. Let us consider along with C the family $T_1(t)$ of its tangent lines and the family $T_2(t)$ of its osculating planes. Under suitable assumptions‡ the intersection $S_2 \cdot T_1(t)$ is a point $P^1(t)$, which generates a (plane) curve C^1. Each tangent $T_1^1(t)$ to C^1 cuts S_1 in a point $P^2(t)$. Now each intersection $S_1 \cdot T_2(t)$ is a point $Q^2(t)$. How are $P^2(t)$ and $Q^2(t)$ related?

The answer is obtained analytically. We give only the outline, since the details are simple computations. Points of the line $T_1(t)$ are given by linear combinations

$$\lambda f_i(t) + \mu f_i'(t),$$

whence $P^1(t)$ has coordinates

$$g_i(t) = \begin{vmatrix} f_0(t) & f_0'(t) \\ f_i(t) & f_i'(t) \end{vmatrix} \qquad (i = 0, \cdots, 3).$$

Similarly $P^2(t)$ has coordinates

$$h_i(t) = \begin{vmatrix} g_1(t) & g_1'(t) \\ g_i(t) & g_i'(t) \end{vmatrix} = \begin{vmatrix} \begin{vmatrix} f_0(t) & f_0''(t) \\ f_1(t) & f_1'(t) \end{vmatrix} & \begin{vmatrix} f_0(t) & f_0'(t) \\ f_1(t) & f_1'(t) \end{vmatrix} \\ \begin{vmatrix} f_0(t) & f_0'(t) \\ f_i(t) & f_i'(t) \end{vmatrix} & \begin{vmatrix} f_0(t) & f_0''(t) \\ f_i(t) & f_i''(t) \end{vmatrix} \end{vmatrix},$$

the second equality following by direct substitution and use of the formula for differentiation of determinants. Now the coordinates of $Q^2(t)$ are found to be

† G. Kowalewski, Determinanten-Theorie, Leipzig, 1909, pp. 83–88.

‡ The hypotheses in Section 3 are sufficient.

$$h_i^*(t) = \begin{vmatrix} f_0(t) & f_0'(t) & f_0''(t) \\ f_1(t) & f_1'(t) & f_1''(t) \\ f_i(t) & f_i'(t) & f_i''(t) \end{vmatrix}.$$

Applying (2) to $h_i^*(t)$ (using $m=2$), we have

$$f_0(t)h_i^*(t) = h_i(t),$$

whence $Q^2(t)=P^2(t)$, since their coordinates are proportional.

The result just obtained may be regarded as a proof of a theorem on developable surfaces. The tangents $T_1(t)$ generate the developable surface σ of C. Since $P^1(t)$ is on $T_1(t)$, C^1 is a plane curve on σ crossing all generators. The tangent plane to σ at $P^1(t)$ is the plane of $T_1(t)$ and $T_1^1(t)$, or, since $P^2(t)$ is on $T_1^1(t)$, it is the plane of $T_1(t)$ and $P^2(t)$. But $T_2(t)$ contains $T_1(t)$ and $Q^2(t)=P^2(t)$, whence $T_2(t)$ is the tangent plane. Since by a change of coordinate system, S_2 can be chosen arbitrarily, $P^1(t)$ is an arbitrary point of $T_1(t)$. Thus the tangent plane to σ is the same along a generator, being the osculating plane of C at the point of tangency of the generator.

3. The generalization. Let a curve C in n-dimensional projective space be generated by $P(t)$ with coordinates $f_i(t)$ $(i=0, \cdots, n)$. The values of t are supposed limited to a region in which the $f_i(t)$ are of class $C^{(n)}$ and all first principal minors of the Wronskian matrix

$$W(t) = (f_i^{(j)}(t);\ i, j = 0, \cdots, n)$$

are not zero. The meaning of this hypothesis will appear presently. Consider spaces S_k $(k=1, \cdots, n-1)$ given by

$$S_k: \qquad x_0 = x_1 = \cdots = x_{n-k-1} = 0.$$

The osculating k-spaces of C are denoted by $T_k(t)$ $(k=1, \cdots, n-1)$. We may define two sequences of curves, C^h, D^h $(h=0, \cdots, n-1)$ as follows. The hypothesis on $W(t)$ may be shown to be equivalent to the proposition that $S_{n-h}\cdot T_h(t)$ is a point $Q^h(t)$ and that $Q^h(t)$ is not in S_{n-h-1}. In fact, $Q^h(t)$ has coordinates $y_i(t)$, where

$$(3) \qquad y_i(t) = \begin{vmatrix} f_0(t) & f_0'(t) & \cdots & f_0^{(h)}(t) \\ \cdots & \cdots & \cdots & \cdots \\ f_{h-1}(t) & f_{h-1}'(t) & \cdots & f_{h-1}^{(h)}(t) \\ f_i(t) & f_i'(t) & \cdots & f_i^{(h)}(t) \end{vmatrix}.$$

Now D^h is the curve generated by $Q^h(t)$; $D^0=C$. The curves C^h are defined inductively. First define $C^0=C$. If C^h has been defined, is generated by $P^h(t)$ and is contained in S_{n-h} $(h=0, \cdots, n-2)$, then $S_{n-h-1}\cdot T_1^h(t)$ is a point $P^{h+1}(t)$

where $T_1^h(t)$ is the tangent line to C^h at $P^h(t)$. We shall establish by induction simultaneously the existence of the points $P^h(t)$ and the fact that $P^h(t)=Q^h(t)$.

The result is obvious for $h=0$. Let $h=0, \cdots, n-2$, and suppose the result true for h. The coordinates of $P^h(t)$ are proportional to those of $Q^h(t)$, namely those given by (3). Clearly $y_h(t)\neq 0$ by the hypothesis on $W(t)$, whence it follows that $T_1^h(t)\cdot S_{n-h-1}$ is a point $P^{h+1}(t)$, whose coordinates are

$$z_i(t)=\begin{vmatrix} y_h(t) & y_h'(t) \\ y_i(t) & y_i'(t) \end{vmatrix};$$

those of $Q^{h+1}(t)$ are

$$w_i(t)=\begin{vmatrix} f_0(t) & f_0'(t)\cdots f_0^{(h+1)}(t) \\ \cdots & \cdots\cdots\cdots \\ f_h(t) & f_h'(t)\cdots f_h^{(h+1)}(t) \\ f_i(t) & f_i'(t)\cdots f_i^{(h+1)}(t) \end{vmatrix}.$$

Applying Sylvester's theorem (2) to $w_i(t)$ (using $m=h+1$) we see that

$$\Delta w_i(t)=z_i(t),$$

where

$$\Delta=\begin{vmatrix} f_0(t) & \cdots f_0^{(h-1)}(t) \\ f_{h-1}(t)\cdots & f_{h-1}^{(h-1)}(t) \end{vmatrix}.$$

Since $\Delta\neq 0$, it follows that $P^{h+1}(t)=Q^{h+1}(t)$. Hence $C^h=D^h$ for $h=0, \cdots, n-1$.

We may sum up our results in the

THEOREM. *Let in a projective space S_n an increasing sequence of subspaces*

$$S_1\subset S_2\subset\cdots\subset S_{n-1}\subset S_n$$

be given. Let α_{hk} be the operation of passing from a curve C in S_k (but not in S_{k-1}) to a curve in S_h $(h<k)$ by intersecting S_h with the $(k-h)$-dimensional osculants of C. If C is a curve in S_n for which each $\alpha_{hn}C$ exists and is not in S_{h-1}, then

$$\alpha_{hh_1}\alpha_{h_1h_2}\cdots\alpha_{h_rn}C=\alpha_{hn}C, \tag{4}$$

where $h<h_1<\cdots<h_r<n$.

By our analysis,

$$\alpha_{hn}C=\alpha_{h,h+1}\alpha_{h+1,h+2}\cdots\alpha_{n-1,n}C,$$

and by induction the left member of (4) is shown to have the same value.

CLUBS AND ALLIED ACTIVITIES

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Michigan State College, East Lansing, Michigan.

A NOTE ON SCALES OF NOTATION

H. D. LARSEN, University of New Mexico

If $1+i$ is taken as the base of a number system, the resulting representations bear some interesting relationships to the mathematics of finance. A number written in the scale of $1+i<2$ properly makes use of two digits, 0 and 1. The positional value of each digit is determined by the definition,

$$a_n a_{n-1} \cdots a_1 a_0 . a_{-1} a_{-2} \cdots a_{-m} = \sum_{j=-m}^{n} a_j (1+i)^j, \qquad a_j = 0 \quad \text{or} \quad 1.$$

As a matter of notation, let the index (k) above a digit indicate that the digit is repeated k times; thus, $1^{(2)}0^{(3)}.1^{(2)} = 11000.11$. The following relationships involving the four fundamental compound interest functions result immediately:

$$(1) \qquad (1+i)^n = 10^{(n)};$$

$$(2) \qquad (1+i)^{-n} = .0^{(n-1)}1;$$

$$(3) \qquad s_{\overline{n}|} = 1^{(n)};$$

$$(4) \qquad a_{\overline{n}|} = .1^{(n)}.$$

The representation of numbers in the scale of $1+i$ effects a novel method of proving many of the theorems in the mathematics of finance, making use of the various properties of such a number system. In particular, it should be noted that a number in the scale of $1+i$ is multiplied by $(1+i)^n = 10^{(n)}$ if the separatrix is moved n places to the right, and is multiplied by $(1+i)^{-n} = .0^{(n-1)}1$ if the separatrix is moved n places to the left. One example should suffice to illustrate the method of proof:

$$\begin{aligned} s_{\overline{n}|}(1+i)^m &= 1^{(n)} \times 10^{(m)} = 1^{(n)}0^{(m)} \\ &= 1^{(n+m)} - 1^{(m)} = s_{\overline{n+m}|} - s_{\overline{m}|}. \end{aligned}$$

This method of proof may have value in suggesting different proofs for certain theorems in the mathematics of finance. Thus, through this device the author developed an elegant proof for the formula,

$$a_{\overline{1}|} + a_{\overline{2}|} + \cdots + a_{\overline{n}|} = (n - a_{\overline{n}|})/i.$$

Further relationships might be mentioned. Circulating decimals in the scale

of $1+i$ are connected with the theory of perpetuities, while the usual algorithms for converting $s_{\overline{n}|}$ and $a_{\overline{n}|}$ from the scale of $1+i$ to the decimal scale are readily identified with the sinking-fund and amortization schedules. Additional relationships are left for the interested reader to discover for himself.

CLUB REPORTS 1942–43

Pi Mu Epsilon, Washington University

Due to unsettled conditions at Washington University the regular monthly meetings for the presentation of papers were dispensed with. The annual banquet and initiation of new members was held on December 13, 1942, when twenty-nine candidates were initiated. An address was heard on the subject:

Geophysical prospecting, by Professor S. H. Van Wambeck of the Electrical Engineering Department.

Officers for the year were: Director, James Embree; Vice-Director, Cecilia Lehman; Secretary, Kenyon Hammack; Treasurer, Professor Eugene Stephens; Corresponding Secretary, Dr. Jessica Young Stephens.

Kappa Mu Epsilon, Albion College

Each of the regular monthly meetings opened with a roll call, requiring those present to name in turn: a formula in calculus (November 10), a mathematician and his outstanding work (December 8), a theorem in geometry (January 12), a formula in trigonometry (March 9), and a formula in analytic geometry (April 13). Titles for the first three meetings were:

Number bases, by Richard Hadley.

The mil and its applications, by Alice Gibb.

Development of symbols of fundamental operations, by Betty Davis.

The topic following the initiation of new members on February 9 was:

Nomographic charts, by Dr. Ingalls.

At the March meeting several five minute talks were given by new members:

Early American arithmetics, by Virginia Tripp.

Mathematics in war, by Russell Miers.

Calculus of Newton, by Barbara Manley.

Napier's logarithms, by Betty Hossfeld.

Mathematics in war, by John Barcroft.

Algebraic congruences, by Helen Shephard.

In April, there was a round table discussion entitled

Science and Mathematics in War, including: *Mathematics*, by Dale Buerstetta; *Physics*, by Earl Dinger; *Chemistry*, by Howard Sherman.

The annual picnic was held in May. Officers for the year were: President, Clare Stanford; Vice-President, James Hollingsworth; Secretary-Treasurer, Elizabeth Davis.

DISCUSSIONS AND NOTES

Edited by MARIE J. WEISS, Sophie Newcomb College, New Orleans, La.

The department of Discussions and Notes is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

RECIPROCAL EQUATIONS

W. V. LOVITT, Colorado College

The following is a simple proof of a formula given by Prof. L. E. Dickson.* The proof requires a knowledge of the theorem in finite differences that the rth difference of the polynomial

$$n^r + a_1 n^{r-1} + \cdots + a_r$$

is $r!$.

If we put $x+x^{-1}=z$ and $x^p+x^{-p}=V_p$, we have the well known recursion formula $V_{n+1}=V_n z - V_{n-1}$. We have†

$$\begin{aligned}
V_1 &= z \\
V_2 &= z^2 - 2 \\
V_3 &= z^3 - 3z \\
V_4 &= z^4 - 4z^2 + 2 \\
V_5 &= z^5 - 5z^3 + 5z \\
V_6 &= z^6 - 6z^4 + 9z^2 - 2 \\
V_7 &= z^7 - 7z^5 + 14z^3 - 7z \\
V_8 &= z^8 - 8z^6 + 20z^4 - 16z^2 + 2 \\
&\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

The recursion formula establishes that, neglecting signs, the first differences of the coefficients in any vertical column give the coefficients in the preceding column. All of the coefficients in the first column are $+1$. Hence, the coefficients of the $(r+1)$th terms of the V_n $(n>0)$ are a series whose rth differences are unity. Obviously

$$V_n = z^n - nz^{n-2} + \cdots + (-1)^r P_r z^{n-2r} + \cdots,$$

where P_r stands for a polynomial in n of degree r.

* L. E. Dickson, Elementary Theory of Equations, p. 83, John Wiley, 1941. For another form of the coefficients in this formula see an article by S. F. Bibb, this MONTHLY, May, 1943, p. 318.

† Lovitt, Elementary Theory of Equations, p. 88, Prentice Hall, 1939.

It remains only to construct these polynomials of degree r so that their rth differences are $+1$, and for the specified values of n the coefficients in column $r+1$ are reproduced. Since the rth difference of P_r is $+1$, we have

$$P_r = (n^r + a_1 n^{r-1} + \cdots + a_r) \div r!.$$

It remains to determine the coefficients a_i. This is accomplished by solving r linear equations in r unknowns.

Let us determine P_3. We have for

$$\begin{aligned} n = 6: &\quad 6^3 + 36a_1 + 6a_2 + a_3 = 3!(-2), \\ n = 7: &\quad 7^3 + 49a_1 + 7a_2 + a_3 = 3!(-7), \\ n = 8: &\quad 8^3 + 64a_1 + 8a_2 + a_3 = 3!(-16). \end{aligned}$$

Whence

$$P_3 = \frac{1}{3!}(n^3 - 9n^2 + 20n) = \frac{n(n-4)(n-5)}{3!}.$$

In like manner

$$P_2 = \frac{n(n-3)}{2!}.$$

To an observant student the pattern should now be clear. The numerator for P_r contains r factors the first of which is n. Omit the next r numbers namely $(n-1)\ (n-2) \cdots (n-r)$. We have

$$V_n = z^n - nz^{n-2} + \frac{n(n-3)}{2} z^{n-4} - \frac{n(n-4)(n-5)}{3!} z^{n-6} + \cdots$$

$$+ (-1)^r \frac{1}{r!} n[n-(r+1)][n-(r+2)] \cdots [n-(2r-1)] z^{n-2r} + \cdots .$$

To obtain the coefficients in column $r+1$, use the coefficient of z^{n-2r} and begin the computation with $n=2r$. The proof is completed by induction. In this induction process one must take note of the following: the first difference of P_r is not P_{r-1} but

$$\frac{1}{(r-1)!}(n-1)[(n-1)-r][(n-1)-(r+1)] \cdots [(n-1)-(2r-3)]$$

which is, however, what P_{r-1} becomes if n is replaced by $n-1$. This will reproduce the preceding column (rth column) of coefficients if we start the computation with an integer n such that $n-1=2r$.

THE ROOTS OF A CAYLEY NUMBER

ROY DUBISCH, Montana State University

It has long been known that the three systems of the complex numbers, the quaternions, and the Cayley numbers possess many points of similarity in spite of the fact that the complex numbers form an associative and commutative system, the quaternions an associative but non-commutative system, and the Cayley numbers a non-associative and non-commutative system. Furthermore, these three algebras are unique among all algebras in that they are the only algebras for which the norm is a sum of squares and the norm of a product equals the product of the norms of the factors.* Thus, there is some justification in showing that the method of De Moivre for finding roots of complex numbers which has been extended by Brand† for finding roots of quaternions may also be used without any serious modification for finding roots of Cayley numbers.

Cayley defined his algebra in terms of eight units $1, i_1, \cdots, i_7$ such that $i_1^2=-1, \cdots, i_7^2=-1$,

$$i_1i_2 = i_3 = -i_2i_1, \qquad i_2i_3 = i_1 = -i_3i_2, \qquad i_3i_1 = i_2 = -i_1i_3,$$

and six similar sets of six relations with 1, 2, 3 replaced by 1, 4, 5; 6, 2, 4; 6, 5, 3; 7, 2, 5; 7, 3, 4; 1, 7, 6; respectively. We may then proceed as Brand did to write any Cayley number, $Z=a_0+a_1i_1+\cdots+a_7i_7$ as

$$Z = k(\cos\theta + \epsilon \sin\theta), \qquad (0 \leqq \theta < 2\pi),$$

where $k=\sqrt{a_0^2+\cdots+a_7^2}$,

$$\cos\theta = \frac{a_0}{k}, \qquad \sin\theta = \frac{\sqrt{a_1^2+\cdots+a_7^2}}{k},$$

and, in case $a_1^2+\cdots+a_7^2\neq 0$, ϵ is the unit vector

$$\epsilon = \frac{a_1i_1+\cdots+a_7i_7}{\sqrt{a_1^2+\cdots+a_7^2}}$$

while, in the contrary case, ϵ may be chosen at pleasure.

We may then verify that $\epsilon^2=-1$ so that one may proceed exactly as for the quaternion problem to obtain the

THEOREM: *A non-real Cayley number has exactly n nth roots. If Z is real and positive, it has just two square roots, $\pm\sqrt{Z}$; in all other cases a real Cayley number has infinitely many roots.*

Finally, we may note that the roots of a Cayley number may be computed by the use of formulas (4) and (5) of Brand's paper.

* See, for example, L. E. Dickson, Quaternions and their generalizations and a history of the eight square theorem, Annals of Mathematics, Vol. 20, 1919, pp. 155–171.

† This MONTHLY, Vol. 49, 1942, pp. 519–520.

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University, and H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editors of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

Plane and Spherical Trigonometry. Alternate Edition. By L. M. Kells, W. F. Kern, and J. R. Bland. New York & London, McGraw-Hill Book Company, Inc., 1943. 17+400 pages $2.00 (with tables $2.75).

Two editions of the same title and a smaller volume entitled *Spherical Trigonometry with Naval and Military Applications* by the same authors have been published and the reviews have appeared in this MONTHLY.*

All the good things that were written about those volumes are emphatically true about this one and will not be repeated here. The authors have naturally drawn upon their preceding work for many exercises and problems, but have added a substantial number of new and up-to-date ones in this edition. The reader will not fail to notice that the definitions of terms used in formulating the problems have been made with unusual care and clarity. The present volume also surpasses its predecessors, and I might add its competitors in the field, in the abundance of well drawn figures to accompany many of the problems to be solved by the students. Some teachers will welcome the early introduction to the use of the slide-rule and the more frequent reference to that technique in the rearrangement of the topics. In view of the apparent strength in the development of the numerical and computational aspect of trigonometry, I don't understand why sec. 28 on accuracy of computation in the previous edition was left out of this one. Even if that topic is discussed at length in the class room, most students want a correct and authoritative statement of it in their mathematics textbook.

The analytic aspect of trigonometry is on the whole well presented. Again from the abundant sets of equations, identities, and other types of problems the instructor can choose enough and of sufficient difficulty to condition the mental sinews of his best students. In common with practically all other trigonometry textbooks this one contains such statements as $\tan 90° = \infty$, which are not true, in preference to tan 90° does not exist, which is true. Also, geometrically minded instructors will regret that the authors did not make use of line values of trigonometric functions, at least in preparation of graphs.

These criticisms detract very little from the all around excellence of the book, which is undoubtedly one of the very best in the field.

GEORGE SAUTÉ

* First Edition and Second Edition reviewed by J. M. Feld in this MONTHLY, vol. 43, 1936, p. 39 and vol. 47, 1940, p. 703. *Spherical Trigonometry with Applications* was reviewed by G. L. Walker in vol. 50, 1943, p. 198.

Engineering Problems Illustrating Mathematics. A project of the mathematics division of the Society for the Promotion of Engineering Education. By J. W. Cell. New York and London, McGraw-Hill, 1943. 11+172 pages. $1.75.

Here is a little book that every teacher of mathematics from college algebra through calculus would do well to have on his desk. The book is a collection of problems comparable in difficulty with the standard problems in the ordinary American textbooks of mathematics, but couched in the language of engineering and physics. It is not intended that the book be put in the hands of all students. In fact the compilers take a most sensible view and state "This collection will be misused if there is any attempt to teach junior or senior engineering concepts in the freshman or sophomore mathematics class room." On the other hand, the collection will also be misused if it leads the student to believe that mathematics is studied only for its utility in solving engineering problems.

Almost all of the problems have been stated in such a way that no understanding of physical terms is necessary for the mathematics involved. After all, should not an engineer be able to slough off all superfluous structure and obtain a mathematical problem which he can attack unhampered by his physical interpretation of each symbol? This book will certainly help the student to develop such an ability. There is of course the danger that a student will be fooled into thinking he knows (or thinking he can fool someone else into thinking he knows) some engineering just because he can glibly use such terms as ohms, henrys, potential energy, or shear curve.

The accuracy of statement and numerical work were checked by sampling and found to be quite satisfactory. Many problems do not give all of the units involved but this is good engineering practice. The five parts are labeled Algebra, Trigonometry, Analytic Geometry, Differential Calculus, and Integral Calculus, respectively. Answers are given to all problems. The figures are well drawn and the printing is excellent. Thanks for a mathematics book printed in war time without a trace of war flavor.

J. F. RANDOLPH

Lessons in Elementary Analysis. By G. S. Mahajani. Third Edition. Poona, India. Aryabushan Press, 1942. 13+298 pages.

The second edition (1931) of this book was reviewed in the MONTHLY by H. R. Cooley in May 1938. The essential features of that edition have been retained. The following changes have been introduced, in addition to alterations in the exercises at the end of each chapter.

The proof of the second mean value theorem for integrals is now that of Landau. Notes on Frullani integrals and on the general form of the remainder in the Taylor series are added. The chapters on infinite integrals and on uniform convergence are rewritten as Chapters X (pp. 185–213) and XI (pp. 214–238). The excellence of the style and of the appearance of the book has been retained.

VIRGIL SNYDER

Calculus. By L. M. Kells. New York, Prentice-Hall, Inc., 1943. 7+509 pages. $3.75.

This is the kind of book any one who knows Professor Kells would expect from him. It has been written with care and with considerable pedagogical insight. There is an abundance of instructive illustrative examples which add much to the value of the book as a text for the student beginning the study of the Calculus. They give him an insight into the wide range of problems whose solution depends upon the methods of the Calculus and stimulate his interest.

In addition to the text of 361 pages there is a frontispiece showing Sir Isaac Newton experimenting with the solar spectrum, a table of integrals, four-place tables of common and natural logarithms, a table of trigonometric functions and one of values of e^x and e^{-x}.

There are a number of matters of detail which should be mentioned. On page 97 in discussing Newton's Method of approximating a root of an equation the author says that the tangent to the curve will cut the x-axis at a point whose abscissa is nearer a root than the abscissa of the point of contact. This statement requires some qualification. The statement on page 106 that certain inequality signs should be reversed if θ is negative is wrong. The reference in the footnote on page 152 to a hyperbola should be to an equilateral hyperbola. In the first line of page 162 the word *differential* should be replaced by the word *infinitesimal*.

In the proof of Rolle's Theorem on page 187 it is not necessary to assume the existence of a derivative at the ends of the interval. The same remark applies to the proof of the law of the mean. The subscript of y_i in the last line of page 191 should be deleted. The form in the heading of §74, page 194, should be $0 \times \infty$, and not 0×0. In §75 there should be some explanation, not necessarily a proof, of the statement that $\lim_{x \to a} (\log y) = \log (\lim_{x \to a} y)$. In the inequalities at the top of page 228 the middle term should be x. In line 3, page 229, the statement should be "s represents the sum of the rectangles with their upper bases below or on the curve." As to line 15, page 230, it should be observed that the law of the mean does not require that the derivative of the function be continuous.

There is a little ambiguity in the definition of work in §100 since it is not explicitly stated that the motion is due exclusively to the force in question. There is a good discussion of the tangent plane to a surface on pages 359 and 360, but a poor definition of the envelope to a one-parameter family of plane curves on page 362. An equality sign is omitted in equation (10), page 374. It is by no means obvious as implied on page 375 that a double integral is equal to an iterated integral. Although no formal proof of this has a place in an introductory text attention should be called to the fact that a proof is necessary. On page 378 reference is made to the distance between two points when clearly it is the square of this distance that is meant.

There is no definition of the area of a curved surface in §156 although a formula is derived for this area. In the definition of convergence of an infinite series on page 400 the word "unique" should be deleted, since if there is a limit

it is unique. The symbol "$<$" in line 10 from the bottom of page 409 should be replaced by the symbol "$\leqq$," and in line 3, page 411, k should be restricted to being a positive integer. The error mentioned in line 6, page 423 should be designated by ϵ, and not by e. The student will have difficulty in seeing the reason for the statement on line 15, page 433, that neither of the values in question gives a maximum or a minimum value of $f(x)$, and the reason for the statement on lines 12 and 13, page 454, is not obvious.

I have here pointed out a number of things that seem to me to need correction. But many of these are due to obvious misprints and even those that are not can be easily corrected in a second printing, or in the class room by an alert teacher. As it stands the book is an unusually good one and when and if these changes are made it will be still better.

W. B. FITE

Grösse, Masszahl und Einheit. By Max Landolt. Zürich, Rascher & Cie., 1943. 85 pages.

This booklet is concerned with the theory of computations involving *physical magnitudes*, *measure numbers*, and *physical units*. Part I (pp. 11–34) consists mainly of a collection of typical numerical computations, such as might be included in a well-written college physics text. Part II (pp. 37–85) is a formalization of such computations in the notation of abstract groups. Magnitudes of the same kind are considered to combine "intensively" to form "intensive" groups (*e.g.*, 2 *gm.*+3 *gm.*=5 *gm.*). Magnitudes of different kinds are considered to combine "qualitatively" to form a "qualitative" group (*e.g.*, a certain mass×a certain acceleration=a certain force). The theory is developed that physical computations consist of combinations of these two types of group-operations.

Part I shows clearly the practical significance of *units* as they are involved in physical computations. Part II begins with a brief discussion of the conceptual difficulties inherent in the operations of multiplying and dividing unlike quantities, and concludes with an interpretation of the group-operations as operations of arithmetic. Apart from the pedagogic value of Part I, the booklet is largely of theoretic interest.

C. C. TORRANCE

Navigational Trigonometry. By P. R. Rider and C. A. Hutchinson. New York, Macmillan Company, 1943. 232 pages. $2.00.

This book, a revision and expansion of an earlier text, is an outgrowth of the war need for more materials on surface and aerial navigation. The text is replete with definitions, problem materials, and methods of solution. It is not a book of theory but of practice. Its illustrations are sufficient and clear. With its practical problems, its five place tables of logarithmic haversines should make it a very usable text in navigational trigonometry.

WILLIAM MELCHER

Elementary Statistical Methods. By Helen M. Walker. New York, Henry Holt and Co., 1943. 25+368 pages. $2.50.

This is a textbook on statistics written for students with extremely meager mathematical backgrounds. It is quite generally believed by mathematicians that people who do not understand mathematics cannot be trusted to apply it and there is little doubt that an understanding of the derivations tends to produce a much more intelligent use of the results. However the fact remains that people with inadequate mathematical training are going to use statistics and the most we can hope is that these people will receive the best possible training consistent with their abilities. Thus the question is whether this textbook is a competent piece of work for what it attempts to do. In the opinion of the reviewer the answer is decidely in the affirmative. The author herself evidently has a clear understanding of statistical theory but she is also capable of taking the point of view of the students who cannot follow the various mathematical derivations.

The amount of mathematics contained in this book is about average for a first course in statistics. There is the difference, however, that the more difficult developments appear in the appendix. If the student is capable of reading the appendix, he is encouraged to do so. Otherwise he is advised to accept the results on authority. It seems likely that the mathematically trained students would attack the appendix with added zest after having seen the results and their applications.

The following list of topics treated in the book will give some idea of the scope: significant digits, class interval, frequency distribution, percentiles, averages, measures of variability, skewness, kurtosis, moments, the normal distribution, regression, correlation, sampling, randomness, and biased sampling. In the development of these topics the emphasis is placed on interpretation rather than on computation or methods of computing. The author makes the following comment. "Confronted with a set of raw data, the novice is tempted to ask himself, 'What measures have I learned to compute which I might make use of in respect to these data?' This is exactly the wrong way around. He ought to say, 'For what questions would I like to have these data provide the answers?'" Following this comment is a list of questions which might be asked of a specific set of data. The development of the methods of answering such questions constitutes the major portion of the book.

The reviewer feels that the mathematical theory of sampling has been somewhat slighted and that additional space in the appendix might well be devoted to this subject. One further criticism. The following statement on page 271 is misleading. "It can be proved mathematically that the sampling distribution of the mean is characteristically normal, and the frequency distribution of any observed set of means obtained from random samples of uniform size will tend toward the normal if enough samples are taken. Note that it is the *number of samples* which must be increased in order to produce this result, not the *number of cases in a single sample.*" The author refers to samples of fifty taken from a

large but finite parent population. Under such circumstances the normal distribution cannot be obtained as a limit by increasing the number of samples. Moreover the size of the samples does affect the degree of approximation to normality.

These criticisms however are minor. On the whole the book is very readable, it is intelligently written and should prove to be a useful text.

A. H. COPELAND

NEW BOOKS RECEIVED

Analytic Geometry. By K. B. Patterson and A. O. Hickson. New York, Crofts, 1944. 187 pages. $2.10.

An Introduction to Navigation and Nautical Astronomy. By W. G. Shute, W. W. Shirk, G. F. Porter, and Courtney Hemenway. New York, The Macmillan Company, 1944. 14+457 pages. $4.50.

Basic Air Navigation. By E. F. Blackburn. New York, McGraw-Hill Book Company, Inc., 1944. 7+300 pages. $3.00.

Basic Mathematics for War and Industry. By P. H. Daus, J. M. Gleason, and W. M. Whyburn. New York, The Macmillan Company, 1944. 11+277 pages. $2.00.

Compact Tables. Five Place Logarithms. Second Edition. By J. P. Ballantine. Seattle, Ballantine, 1944. $0.50.

Logarithms of Numbers. Trigonometric Functions. Compact Tables. By J. P. Ballantine. Seattle, Ballantine, 1944. $0.25.

Mathematics for Navigators. By Delwyn Hyatt and B. M. Dodson. New York and London, McGraw-Hill Book Company, Inc., 1944. 7+106 pages. $1.25.

"Student's" Collected Papers. Edited by E. S. Pearson and John Wishart with a foreword by Launce McMullen. London, University College, Biometrika office, Cambridge University Press, 1943. 14+224 pages. 15s.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 24 Strathearn Boulevard, Toronto 10, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 621. *Proposed by C. D. Olds, Purdue University*

Show directly (*i.e.*, without substituting θ into the value given by the standard algebraic proofs) that

$$\frac{1}{a}+\frac{1}{a}+\cdots=e^{-\theta},$$

where a is a positive number and $\theta=\operatorname{arg\,sinh}(a/2)$.

E 622. *Proposed by V. Thébault, San Sebastián, Spain*

Find the smallest four-digit number such that the sum of products of pairs of digits is equal to the sum of products of sets of three.

E 623. *Proposed by N. A. Court, University of Oklahoma*

The circumsphere of a tetrahedron $ABCD$ meets four "cevians" LA, LB, LC, LD in the points A', B', C', D'; $A''B''C''D''$ is a tetrahedron homothetic to the tetrahedron $A'B'C'D'$ with respect to the point L. If P and Q are any two points in space, show that the four spheres $A'A''PQ$, $B'B''PQ$, $C'C''PQ$, $D'D''PQ$ are coaxal.

E 624. *Proposed by D. H. Browne, Buffalo, N. Y.*

Show that the integer nearest to $n!/e$ is a multiple of $n-1$.

E 625. *Proposed by C. A. B. Smith, Trinity College, Cambridge, England*

Let X_i, Y_i, Z_i be the cofactors of x_i, y_i, z_i in the general third-order determinant

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = D.$$

(*Cf.* E 481 [1942, 257–260].) Prove that

$$\begin{vmatrix} X_2X_3 & Y_2Y_3 & Z_2Z_3 \\ X_3X_1 & Y_3Y_1 & Z_3Z_1 \\ X_1X_2 & Y_1Y_2 & Z_1Z_2 \end{vmatrix} = -D^2 \begin{vmatrix} x_2x_3 & y_2y_3 & z_2z_3 \\ x_3x_1 & y_3y_1 & z_3z_1 \\ x_1x_2 & y_1y_2 & z_1z_2 \end{vmatrix}.$$

SOLUTIONS

Placing the Last Digit First

E 584 [1943, 454]. *Proposed by C. O. Oakley, Haverford College*

(a) Find the smallest integer N such that, if the units digit is transposed from right to left, a number M is obtained where $M=5N$.

(b) Find the largest integer which satisfies these conditions (without complete repetition of digits).

Solution by R. K. Allen, Montpelier, Vermont. This question has been completely answered by W. B. Chadwick [1941, 251–253]. The smallest number is $(10^6-1)/7=142857$, and the largest is

$$9(10^{42}-1)/49 = 183673469387755102040816326530612244897959,$$

these being the repetends of the decimal equivalents of 1/7 and 9/49, respectively.

Also solved by W. E. Buker, J. E. Eaton, Irving Kaplansky, S. Karlin, Walter Penney, Milton Schiffenbauer, E. P. Starke, and the proposer.

A Quartic Equation

E 588 [1943, 512]. *Proposed by J. Rosenbaum, Bloomfield, Connecticut*

Solve the equation

$$x^4 - 2cx^3 + (2c^2 - r^2)x^2 + 2cr^2x - c^2r^2 = 0.$$

When will two of the roots be rational? Can all four roots be rational?

Solution by E. P. Starke, Rutgers University. The proposed equation can be written at once

$$(x^2 - cx + c^2)^2 = (r^2 + c^2)(x - c)^2. \tag{1}$$

Put

$$s = \sqrt{r^2 + c^2} = (x^2 - cx + c^2)/(x - c), \tag{2}$$

which is rational whenever x is. Then $x^2-cx+c^2=sx-sc$, and

$$x = \tfrac{1}{2}(s + c) \pm \tfrac{1}{2}\sqrt{(s + c)(s - 3c)}, \tag{3}$$

which is rational if and only if $(s+c)(s-3c)=k^2$ is a perfect square. Put $s+c=\rho u^2$, $s-3c=\rho(k/\rho u)^2=\rho v^2$. Then, since $r^2=s^2-c^2$,

$$s - c = \rho(r/\rho u)^2 = \rho w^2.$$

Elimination of s and c from these equations gives

$$u^2 + v^2 = 2w^2, \tag{4}$$

for which the solutions in integers are known, *viz.*,

$$u = \sigma(a^2 + 2ab - b^2), \qquad v = \sigma(a^2 - 2ab - b^2), \qquad w = \sigma(a^2 + b^2). \tag{5}$$

We have, then, by substitution,

$$c = 2\rho\sigma^2 ab(a^2 - b^2), \qquad r = \rho\sigma^2(a^2 + b^2)(a^2 + 2ab - b^2). \tag{6}$$

These are integral values of c and r for which two roots of (1) are rational.

The other two roots of (1) are obtained by using, in place of (2),

$$s = -(x^2 - cx + c^2)/(x - c). \tag{2'}$$

The strictly analogous argument leads to

$$(3') \qquad x = \tfrac{1}{2}(-s+c) \pm \tfrac{1}{2}\sqrt{(s-c)(s+3c)};$$

$$s+c = \rho u^2, \qquad s-c = \rho w^2, \qquad s+3c = \rho z^2;$$

$$(4') \qquad w^2 + z^2 = 2u^2;$$

$$(5') \qquad z = \tau(p^2+2pq-q^2), \qquad w = \tau(p^2-2pq-q^2), \qquad u = \tau(p^2+q^2);$$

$$(6') \qquad c = 2\rho\tau^2 pq(p^2-q^2), \qquad r = \rho\tau^2(p^2+q^2)(p^2-2pq-q^2).$$

If all four roots are to be rational, we must have simultaneously

$$s+c = \rho u^2, \qquad s-c = \rho w^2, \qquad s+3c = \rho z^2, \qquad s-3c = \rho v^2,$$

which give both (4) and (4'). But these equations can be put in the form $2w^2-u^2=v^2$, $2u^2-w^2=z^2$, which imply $w^2=u^2=v^2=z^2$. (See W. T. King, this MONTHLY, vol. 6, 1899, pp. 151–155: a discussion which can be radically simplified.) But then $c=0$, and (1) has the four rational roots $x=0,\ 0,\ r,\ -r$. No other set of four rational roots is possible.

Also solved by Hazel Schoonmaker Wilson and the proposer.

Two Generalizations of the Problème des Rencontres

E 589 [1943, 512]. *Proposed by Lloyd Dulmage, University of Manitoba*

Let ${}_2H_n$ denote the number of ways of putting n letters and n checks into n envelopes (respectively) so that every envelope contains either a wrong letter or a wrong check (or both), and let ${}_2J_n$ denote the number of ways when each envelope contains both a wrong letter and a wrong check. Show that

$${}_2H_n = \Delta^n(0!)^2, \qquad {}_2J_n = (\Delta^n 0!)^2,$$

and generalize to ${}_sH_n$, ${}_sJ_n$.

Solution by John Riordan, Bell Telephone Laboratories. In the general case, this is a problem of enumerating coincidences in sets of permutations of n elements (or packs of cards labelled 1 to n): $s+1$ permutations of n elements are placed under one another, and a count is made of positions in which elements are alike.

The total number of configurations is $n!^s$; for, each of the permutations except a standard one may be selected in $n!$ ways. If ${}_sH_n$ is the number with no coincidences, the number with i concidences is

$$\binom{n}{i}\,{}_sH_{n-i};$$

for, the positions of coincidence may be selected in $\binom{n}{i}$ ways, and the remainder must have no coincidences. Thus

$$n!^s = \sum_{i=0}^{n}\binom{n}{i}\,{}_sH_{n-i} = (1+{}_sH)^n.$$

But

$$n!^s = E^n 0!^s = (1 + \Delta)^n 0!^s,$$

where E and Δ are the usual finite difference operators. Comparing coefficients,

$$_sH_n = \Delta^n 0!^s.$$

In particular, $_1H_n = \Delta^n 0!$. This is the well known solution for the Problème des Rencontres, the number of permutations on any one line having no coincidences with the standard. Hence the number of configurations in which each of the s lines has no coincidences with the standard (though they may coincide with one another) is

$$_sJ_n = (_1H_n)^s = (\Delta^n 0!)^s.$$

It may be noted that, according to Todhunter, the first result has been given by De Moivre, *Doctrine of Chances* (Problem XXXVI, 6th corollary); it also appears in Laplace's *Théorie Analytique des Probabilités* (National Edition, pp. 244–245). The numbers $_sH_n$ have interesting arithmetical properties, such as

$$_sH_{n+p} \equiv -\ _sH_n \pmod{p}, \qquad _{s+p-1}H_n \equiv\ _sH_n \pmod{p},$$

for any prime p. They also satisfy the recurrence relation

$$_sH_{n+1} = (-1)^{n+1} + (n+1)\sum_{i=0}^{s-1}\binom{n}{i}\Delta^i(n+1-i)^{s-1}\,{}_sH_{n-i};$$

e.g.,

$$_1H_{n+1} = (-1)^{n+1} + (n+1)\,{}_1H_n,$$
$$_2H_{n+1} = (-1)^{n+1} + (n+1)^2\,{}_2H_n + n(n+1)\,{}_2H_{n-1}.$$

Two Sequences of Perfect Squares

E 592 [1943, 560]. *Proposed by V. Thébault, San Sebastián, Spain*

Show that the numbers

$$729,\ 71289,\ 7112889,\ 711128889, \cdots,$$
$$7744,\ 97970404,\ 997997004004,\ 9997999700040004, \cdots$$

are perfect squares. Generalize the latter sequence to an arbitrary scale of notation (with base greater than 4).

Solution by E. D. Schell, Arlington, Va. The general term of the first sequence is shown to be a square as follows:

$$\begin{aligned} 7\cdot 10^{2n} + (10^{2n-1} + \cdots + 10^{n+1}) + 2\cdot 10^n + 8(10^{n-1} + \cdots + 10) + 9 \\ &= 7\cdot 10^{2n} + (10^{2n} - 10^{n+1})/9 + 2\cdot 10^n + 8(10^n - 10)/9 + 9 \\ &= (64\cdot 10^{2n} + 16\cdot 10^n + 1)/9 \\ &= \{(8\cdot 10^n + 1)/3\}^2. \end{aligned}$$

For the second, generalized to any radix $r>4$, we have

$$\begin{aligned}(r-1)(r^{4n-1}+\cdots+r^{3n+1})+(r-3)r^{3n}+(r-1)(r^{3n-1}+\cdots+r^{2n+1})&\\ +(r-3)r^{2n}+4r^n+4&\\ =r^{4n}-r^{3n+1}+(r-3)r^{3n}+r^{3n}-r^{2n+1}+(r-3)r^{2n}+4r^n+4&\\ =r^{4n}+r^{2n}+4-2r^{3n}-4r^{2n}+4r^n&\\ =(r^{2n}-r^n-2)^2.&\end{aligned}$$

Also solved by D. H. Browne, Stanley Hughart, E. P. Starke, and the proposer.

The Monge Point of a Variable Tetrahedron

E 593 [1943, 560]. *Proposed by N. A. Court, University of Oklahoma*

A variable tetrahedron has three fixed vertices, a fixed circumsphere, and the sum of the squares of its edges is constant. Find the locus of its Monge point.

Solution by Howard Eves, Syracuse University. (Numbers in parentheses refer to articles in N. Altshiller Court's *Modern Pure Solid Geometry*.) Let A, B, C be the fixed vertices, V the variable vertex, O the center of the fixed circumsphere, S the centroid of triangle ABC, G the centroid and M the Monge point of the tetrahedron. Then, by (187), VS is constant, and the locus of V is a circle Σ, which lies on the circumsphere and has SO for axis. Therefore, by (170), the locus of G is a circle with radius one quarter that of Σ and likewise having SO for axis. Finally, by (230), the locus of M is a circle with radius one half that of Σ and again having SO for axis.

Also solved by J. S. Guérin, L. M. Kelly, and the proposer.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4120. *Proposed by F. J. Duarte, Caracas, Venezuela*

Given k and h as positive integers, show that

$$\sum_{n=0}^{h-1}(-1)^n\frac{(h-1)(h-2)\cdots(h-n)}{n!}\,\frac{k(k+1)\cdots(k+h-n-1)}{(h-n)!}=\binom{k}{h},$$

where for $n=0$ the summand is $k(k+1)\cdots(k+h-1)/h!$.

4121. *Proposed by Alfred Brauer, University of North Carolina*

In generalization of results of Sylvester and Mirimanoff the following theorem was proved, H. F. Baker, *Proc. London Math. Soc.*, (2) vol. 4, 1906, pp. 131–135.

Let d and m be relatively prime positive integers and $mm' \equiv 1 \bmod d$. Denote by $(m'x)$ the least positive residue of $m'x \bmod d$. Then

$$\frac{d^{\phi(m)} - 1}{m} \equiv \sum_x \frac{(m'x)}{m - x} \bmod m,$$

where x runs over the positive integers less than m and relatively prime to m.

Recently, P. Kesava Menon proved the following theorems, *Proc. Indian Academy of Science*, Sect. A, vol. 17, 1943, pp. 107–113.

Let d, m, and n be integers such that I. $dn = m - 1$; II. $dn = m + 1$. Then we have

$$\text{I.} \quad \frac{d^{\phi(m)} - 1}{m} \equiv \frac{1}{d} \sum_x \frac{\{x/n\}}{x} \bmod m;$$

$$\text{II.} \quad \frac{d^{\phi(m)} - 1}{m} \equiv \frac{1}{d} \sum_x \frac{[x/n]}{x} \bmod m;$$

respectively, where x runs over the positive integers less than m and relatively prime to m, and where $\{a\}$ denotes the smallest integer greater than or equal to a, and $[a]$ denotes the greatest integer less than or equal to a.

Prove that these results are special cases of Baker's theorem.

4122. *Proposed by Otto Dunkel, Washington University*

Show that $\Delta^{n-r}0^n/(n-r)!$, where n and r are non-negative integers, $n \geqq r$ is a polynomial $f_r(n)$ in n of degree $2r$. The polynomial $f_r(x)$, $r \geqq 1$, has a positive integral value for all integral values of x, positive or negative, except for $x = 0, 1, 2, \cdots, r-1, r$ for which values it vanishes. If r is an odd integer $\geqq 3$, $r-1$ and r are double roots.

4123. *Proposed by V. Thébault, San Sebastián, Spain*

For what values of m is the product $(2m+1)(10m+1)$ a square of an integer? *Application.* In what systems of numbers is a number of the form *aabb* the square of a number of two digits *bb*?

SOLUTIONS

Partitions

4073 [1943, 125]. *Proposed by J. M. Feld, New York*

Let p_1 be the number of partitions of the positive integer N into an odd number of distinct positive integers and let p_2 be the number of partitions of N into an even number of distinct positive integers. For instance, for $N = 8$, the

odd partitions included in p_1 are 8, 5+2+1, 4+3+1; and the even partitions included in p_2 are 7+1, 6+2, 5+3. Prove that: (1) If $N=n(3n\pm1)/2$, where n is an odd integer, $p_1=p_2+1$. (2) If n is an even integer, $p_1=p_2-1$. (3) If N does not have the above form $p_1=p_2$.

Solution by R. C. Buck, Student, Harvard University. This is simply Euler's theorem on p_1 and p_2. (See Hardy & Wright, *Theory of Numbers*, p. 284, Theorem 358.)

This result follows at once from Euler's Identity

$$(1-x)(1-x^2)(1-x^3)\cdots = 1+\sum_1^\infty (-1)^n\left[x^{n(3n+1)/2}+x^{n(3n-1)/2}\right]$$

for the coefficient of x^N on the left is $p_2(N)-p_1(N)$, while on the right it is 0 unless N is of the form $n(3n\pm1)/2$ when it is $(-1)^n$.

F. Franklin has given a diagrammatic proof of this, reproduced in Hardy-Wright, p. 285.

Solved also by the proposer in a similar manner.

Conic Inscribed in a Triangle

4043 [1942, 408]. *Proposed by H. F. Sandham, Trinity College, Dublin*

Prove that the angle in which the major auxiliary circle of a conic inscribed in a triangle cuts the nine point circle, is equal to the angle which the foci of the conic subtend at the inverse of one of them in the circumcircle. Complete this result and deduce that the minor auxiliary circle of the conic which has the Brocard points as foci, touches the nine point circle, and the major auxiliary circle cuts the latter in an angle which is the complement of three times the Brocard angle of the triangle.

Solution by the Proposer. Taking the radius of the circumcircle (O) as unity the origin at the circumcenter O, the Euler line as the axis of reals, and α, β, γ the complex numbers defining the vertices of the triangle, let s_1, s_2, s_3 denote the elementary symmetric functions of these numbers. If x, y denote a pair of isogonal conjugates,

$$x+y+s_3\bar{x}\bar{y}=s_1, \qquad \bar{x}+\bar{y}+xy/s_3=s_1. \tag{1}$$

The center C of the conic inscribed in the triangle with foci at x, y is $(x+y)/2$ and the radius of the major auxiliary circle (C) is $\rho=|1-x\bar{y}|/2$. The center N of the nine-point circle (N) is $s_1/2$ and its radius is $1/2$. Let θ denote the angle between the circles (N) and (C), then

$$\begin{aligned}|1-x\bar{y}|\cos\theta &= (1-x\bar{y})(1-\bar{x}y)/4+1/4-(x+y-s_1)(\bar{x}+\bar{y}-s_1)/4,\\ &= (1-x\bar{y})(1-\bar{x}y)/4+1/4-x\bar{x}y\bar{y}/4,\\ &= (1-x\bar{y}+1-\bar{x}y)/4.\end{aligned}$$

where (1) has been used in the reduction. Hence

$$1 - x\bar{y} = 2\rho e^{i\theta}, \tag{2}$$

and from this the first part of the problem follows.

Calculating the left member of (2) we find that

$$1 - x\bar{y} = \frac{x^3 - s_1x^2 + s_2x - s_3}{s_3(x\bar{x} - 1)}, \quad \text{or}$$
$$2\rho(x\bar{x} - 1)e^{i\theta} = (x - \alpha)(x - \beta)(x - \gamma)/\alpha\beta\gamma. \tag{3}$$

Hence θ is the sum of the angles subtended at the vertices by the line joining the point x to O, and from this we have the theorem:

The angle between the pedal circle of a point and the nine-point circle is the complement of the sum of the angles each line joining the point to a vertex makes with an adjacent side, no side being taken twice.

From this the second part of the problem follows.

Editorial Note. The value of ρ may be derived for an arbitrary axis of reals through O as follows. Let y_1 be the orthogonal projection of y on the side $\beta\gamma$ of the triangle, then $4\rho^2=(2y_1-x-y)(2\bar{y}_1-\bar{x}-\bar{y})$ which may be expressed in terms of the complex numbers β, γ; and for the other two sides there are similar expressions. Taking the sum of these gives

$$4s_2\rho^2 = 2(s_1^2 - s_2) - 2s_1(x + y) - (s_1s_2 - 3s_3)(\bar{x} + \bar{y}) + 3xy$$
$$+ (s_2^2 - 2s_1s_3)\bar{x}\bar{y} + s_2(x\bar{x} + y\bar{y}). \tag{1}$$

Using the general equations

$$x + y + s_3\bar{x}\bar{y} - s_1 = 0, \qquad s_3(\bar{x} + \bar{y}) + xy - s_2 = 0,$$

we obtain after removing the factor s_2, which is zero only if the triangle is equilateral

$$4\rho^2 = (1 - x\bar{y})(1 - \bar{x}y). \tag{2}$$

This requires long and complicated calculations. Of the two conjugate imaginary factors on the right we may suppose that $1-x\bar{y}$ is that one with a positive angle θ, $0\leqq\theta\leqq\pi$. Then $2\rho e^{i\theta}=1-x\bar{y}$; and if we denote by y' the complex number for the inverse of the point y in (O) we have $y'\bar{y}=1$ and then $2\rho e^{i\theta}=(y'-x)/y'$. This gives the theorem in the first part of the problem.

In the formula (3) of the above solution it should be noticed that, if x is inside (O), the quantity $x\bar{x}-1$ is negative. This formula gives a theorem of the kind stated but it regards the *algebraic* sum of the angles subtended, and it requires considerable care to state the theorem precisely.

If $x\neq y$, it will be seen from the first theorem regarding θ that a necessary and sufficient condition for the tangency of (C) and (N) is that the focal axis xy passes through O. In the limit case $x=y$ we have (C) as the inscribed or an escribed circle. In the two solutions of 3857 [1940, 183] are other proofs of the

theorem that the minor auxiliary circle of the Brocard ellipse is tangent to (N), the nine-point circle of the corresponding triangle.

Limit Functions for Infinite Set of Functions

4058 [1942, 618]. *Proposed by R. P. Agnew, Cornell University*

Give an example of a sequence $f_n(x)$ of real continuous functions, defined over $-\infty < x < \infty$ and vanishing outside the interval $-1 \leqq x \leqq 1$, such that the dominating function $F(x)$ defined by

$$F(x) = \text{l.u.b.}_{n=1,3,2,\cdots} |f_n(x)| \tag{1}$$

and the inferior and superior limit functions

$$\liminf_{n\to\infty} f_n(x), \qquad \limsup_{n\to\infty} f_n(x) \tag{2}$$

are all continuous and such that the members of the inequality (in which integration is over $-\infty < x < \infty$)

$$\begin{aligned} -\int F(x)dx \leqq \int \liminf_{n\to\infty} f_n(x)dx \leqq \liminf_{n\to\infty} \int f_n(x)dx \\ \leqq \limsup_{n\to\infty} \int f_n(x)dx \leqq \int \limsup_{n\to\infty} f_n(x)dx \leqq \int F(x)dx \end{aligned} \tag{3}$$

assume in order the values in the arithmetic progression $-5, -3, -1, 1, 3, 5$.

Remark: An elegant theorem on "integration of sequences" states that if $f_n(x)$ is a sequence of functions measurable over E_m (Euclidean space of m dimensions) and the dominating function (1) is integrable (Lebesgue), then the functions (2) are integrable and (3) holds. See Caratheodory, *Vorlesungen über Reele Funktionen*, Leipzig, 1927, p. 444. An example meeting the conditions of the problem shows that, even when all functions involved are continuous, the differences between successive members of (3) may all be equal to the constant 2.

Solution by B. H. Bissinger, Cornell University. For $k = 1, 2, 3$ let

$$g_k(x) = \begin{cases} 4kx, & 0 \leqq x \leqq \frac{1}{2} \\ -4k(x-1), & \frac{1}{2} \leqq x \leqq 1 \\ 0, & \text{elsewhere} \end{cases}.$$

If we define

$$f_{2n-1}(x) = \begin{cases} g_3(x), & x \geqq 0 \\ -g_2(-x), & x \leqq 0 \end{cases}, \qquad f_{2n}(x) = \begin{cases} -g_1(x), & x \geqq 0 \\ 0, & x \leqq 0 \end{cases}$$

for $n = 1, 2, \cdots$, then for the sequence $f_n(x)$, the members of the inequality (3) assume the required values.

Solved also by C. M. Ablow and the proposer.

Editorial Note. The proposer gave the following example. Let $\phi(x)$ be the function which vanishes when $x<0$ and when $x>1$, and which has for its graph in the interval $0 \leqq x \leqq 1$ of the x, y plane a broken line running from (0, 0) successively through (1/3, 0), (1/2, −3), (2/3, 0), (5/6, 6) to (1, 0). Let $\psi(x) = -\phi(1-x)$ so that the graph $\psi(x)$ in the interval $0 \leqq x \leqq 1$ is the broken line joining in succession (0, 0), (1/6, −6), (1/3, 0), (1/2, 3), (2/3, 0) and (1, 0). Let $f_n(x) = \phi(x)$ when n is odd, and $f_n(x) = \psi(x)$ when n is even.

Ablow gave the sequence defined as follows;

$$f_0(x) = -5(x-1),\ 0 \leqq x \leqq 1: \quad = 5(x+1),\ -1 \leqq x \leqq 0; \quad = 0,\ |x| > 1;$$

$$\begin{aligned} f_{2k-1}(x) &= -(x-1) - 2(x-1)\cos k\pi x, && 0 \leqq x \leqq 1, \\ &= (x+1) + 2(x+1)\cos k\pi x, && -1 \leqq x \leqq 0, \\ &= 0, && |x| > 1 \\ f_{2k}(x) &= -f_{2k-1}(x), && k = 1, 2, 3, \cdots . \end{aligned}$$

He stated that

$$\limsup_{k\to\infty} f_{2k-1}(x) = -3(x-1), \qquad 3(x+1), \qquad 0,$$

$$\limsup_{k\to\infty} f_{2k}(x) = -(x-1), \qquad (x+1), \qquad 0$$

for the respective intervals above. His proofs used the theory of point sets. It may appear simpler to consider first rational values of x, and then for irrational values consider the sequence of the convergents p_i/q_i when x is expressed as a simple continued fraction. We then have an easy proof for lim sup $f_{2k-1}(x)$ as stated above by taking $k = 2q_i$. For lim sup $f_{2k}(x)$ where $x = p/q$ is rational and p is an odd integer, we take k as an odd multiple of q. If x is irrational we consider only those convergents p_i/q_i of x for which p_i is an odd integer, and there is an infinite number of such. Here the limit points lie on the curve defined above. But this is not the case where $x = p/q$ is rational and where p is an even integer. Simple examples of such discontinuities in the limit curve in this case are $x = 0,\ 2/3$.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

Assistant Professor A. V. Baez of Wagner College has been appointed acting instructor in physics at Stanford University.

Associate Professor H. A. Bender of the University of Akron has been appointed to an associate professorship at Rhode Island State College.

Professor Felix Bernstein of New York University has been appointed professor emeritus.

Assistant Professor H. F. Bright of Denison University has been appointed to an assistant professorship at the University of Rochester.

Dr. F. A. Butter, Jr., of the University of Wisconsin has been appointed lecturer in mathematics at Lawrence College.

Visiting Assistant Professor Nancy Cole of Kenyon College has been promoted to a visiting associate professorship.

Associate Professor J. D. Elder of Lynchburg College has been appointed to an assistant professorship in the physics department of Wabash College, Crawfordsville, Indiana.

Dr. R. W. Hamming of the University of Illinois has been appointed to an assistant professorship at the Speed Scientific School of the University of Louisville.

C. B. Helms of Antioch College is now teaching in the Naval Preflight School at the University of Pennsylvania.

Assistant Professor J. F. Heyda of Denison University has been appointed to an assistant professorship at St. Ambrose College, Davenport, Iowa.

Dr. D. H. Hyers of California Institute of Technology has been appointed to an associate professorship at the University of Southern California.

Assistant Professor R. N. Johanson of Hamilton College has been appointed to an assistant professorship at Boston University.

W. S. Krogdahl has been appointed to an adjunct professorship at the University of South Carolina.

Professor H. M. Mac Neille of Kenyon College is now overseas serving in a civilian capacity.

Professor E. J. McShane of the University of Virginia is on leave of absence to serve as head and mathematician of the Exterior Ballistics Branch of the Ballistic Research Laboratory at Aberdeen Proving Ground.

J. D. Novak of MacMurray College has been appointed to an associate professorship at the University of South Carolina.

Professor Irene Price of the State Teachers College, Oshkosh, Wisconsin, has been granted a leave of absence to serve as statistical analyst in the Army Air Forces.

M. A. Scheier of St. Bonaventure College has been appointed dean of the Division of Natural Sciences and Mathematics.

Dr. James Singer of Brooklyn College has been promoted to an assistant professorship.

Professor Emeritus Harris Hancock of the University of Cincinnati died on March 19, 1944. He was a charter member of the Mathematical Association.

SUMMER COURSES

The following institutions announce courses in mathematics for the summer of 1944:

Brown University. From June 12 to September 2 the program of advanced instruction and research in mechanics will be in its fourth year. Since this program is supported by the United States Office of Education, the Carnegie Corporation of New York, and the Rockefeller Foundation, no fees will be charged. The faculty will consist of Professors Stefan Bergman, Lipman Bers, Will Feller, D. L. Holl, Witold Hurewicz, Willy Prager, J. D. Tamarkin, Dr. R. K. Luneberg and one other still to be chosen. Courses in the following subjects are planned: introduction to partial differential equations, geometrical foundations of mechanics, practical analysis, differential and integral equations of mathematical physics, principles of mechanics, elasticity, plasticity, advanced elasticity, fluid dynamics, mathematical optics, and seminars and research in mechanics and other branches of applied mathematics. A special feature will be a course in mathematical optics under the leadership of Dr. R. K. Luneberg of the Spencer Lens Company whose lectures will be supplemented by Dr. Max Herzberger of the Eastman Kodak Company and Professor J. L. Synge.

The Catholic University of America. The following advanced courses will be offered: From June 13 to September 22. By Professor Finan: selected topics in algebra with applications to geometry. By Professor Ramler: functions of a complex variable. From July 3 to August 10. By Professor Rice: advanced calculus. By Professor Finan: theory of equations, selected topics in algebra and number-theory. By Professor Ramler: advanced Euclidean geometry, differential equations, synthetic projective geometry.

Columbia University. From July 3 to August 11 the following graduate courses will be offered: By Professor Kasner: survey of modern mathematics, differential geometry. By Professor Lorch: introduction to higher algebra. By Professor Murray: theory of functions of a complex variable. By Professor Ritt: differential equations.

Cornell University. From June 30 to October 21 in addition to the usual elementary courses, a course in spherical trigonometry and map projections, a course in differential equations, and certain courses continuing from the present term, the following courses will be offered: By Professor Flexner: topology. By Professor Hurwitz; partial differential equations of mathematical physics.

Northwestern University. From June 26 to August 26 the following advanced courses will be offered: definite integrals, solid analytic geometry, theory of statistics, history and teaching of mathematics, fundamental principles of alge-

bra, a seminar in analysis. Those who desire to do so may take any of these courses for six weeks instead of nine weeks, thus leaving on August 5.

The Ohio State University. From June 13 to September 1 the following advanced courses in mathematics will be offered: By Professor Bamforth: advanced calculus, Fourier series and spherical harmonics. By Professor Blumberg: fundamental ideas in algebra and geometry. By Dr. Mickle: solid analytic geometry. By Professor Rado: introduction to the theory of a complex variable, calculus of variations. By Professor Synge: advanced engineering mathematics II, vector analysis.

The State University of Iowa. From June 12 to August 4 the following graduate courses will be offered: By Professor Conkwright: differential equations. By Professor Knowler: elementary theory of statistics. By Professor Oberg: vector analysis. By Professor Price: modern secondary mathematics. By Professor Ward: matrices and determinants. By Professor Woods: pure geometry. There is an independent study unit for graduate students from August 7 to August 25.

The University of Chicago. The first term of the summer quarter is from June 20 to July 29 and the second term from July 31 to September 9. The following advanced courses will be offered: By Professor Albert: Galois theory. By Professor Barnard: metric differential geometry (second term only). By Professor Hartung: two courses in the teaching of mathematics. By Professor Reid: calculus of variations—multiple integrals, differential equations, integral and functional equations. By Professor Schilling: continuous groups.

The University of Illinois. From June 12 to August 5 the following graduate courses will be offered: By Professor Brahana: algebra. By Professor Trjitzinsky: complex variables. By Professor Bourgin: partial differential equations, differential geometry. By Dr. Welker: theory of statistics. From June 12 to September 30 courses will be offered in the following intermediate subjects: differential equations, advanced calculus, elementary statistics, fundamental concepts of mathematics, introduction to higher algebra, and introduction to higher analysis.

The University of Michigan. From July 3 to August 26 the following courses will be offered in addition to the standard courses in differential equations, theory of equations and determinants, and advanced calculus: By Professor Carver: introduction to air navigation. By Professor Craig: theory of statistics I. By Professor Dwyer: theory of statistics II. By Professor Fischer: finite differences. By Professor Hildebrandt: theory of functions of a complex variable, integral equations. By Professor Karpinski: teaching of algebra, history of geometry and trigonometry. By Professor Rainich: dynamics, topics in higher geometry. By Professor Thrall: advanced solid analytic geometry, modern algebra. By Professor Wilder: introduction to the foundations of mathematics, topology.

The University of North Carolina. The following advanced courses will be offered: From June 12 to July 20. By Professor Browne; theory of equations. By Professor Henderson: differential equations. By Professor Winsor: college geometry. From July 21 to August 29. By Professor Garner: history of mathe-

matics. By Professor Henderson: analytic geometry of space. By Professor Lasley: projective geometry.

University of Southern California. From July 1 to August 11 the following advanced courses will be offered: By Professor Ames: theory of probability and statistics. By Professor Hyers: modern higher algebra. By Professor Steed: vector analysis.

The University of Virginia. From July 1 to August 25 the following advanced courses will be offered: By Professor Hedlund: differential equations and applied mathematics. By Professor Whyburn: functions of real variables.

The University of Wyoming. The following advanced courses will be offered: From July 8 to August 5. By Professor Barr: astronomy, college geometry. By Professor Neubauer: advanced algebra. By Professor Varnum: advanced calculus. From August 7 to September 2. By Professor Barr: astronomy, differential equations. By Professor Neubauer: fundamental concepts.

WAR INFORMATION

EDITED BY C. V. NEWSOM

Send news reports upon the utilization of mathematicians or mathematics in war activities to C. V. Newsom, University of New Mexico, Albuquerque, New Mexico.

SALARIES AND TEACHING LOADS IN COLLEGE TRAINING PROGRAMS

Several studies have been made in recent months pertaining to salaries and teaching loads in institutions having Army and Navy College Training Programs. The conclusions reached in two of the studies have been summarized below.

The American Council on Education asked for information on salaries and teaching loads in sixty selected institutions having Army and Navy contracts. The colleges and universities were chosen so as to give a fair sample of participating institutions. Replies were received from 45 colleges and universities, and an analysis of the answers reveals that the following policies are being followed by the majority of institutions:

1. Increases in salary resulting from the change to a twelve-month year are being prorated on the basis of the increased number of months involved. Comparatively few institutions have attempted to develop a formula other than that of proportionate months, but adjustments have been made in individual cases on the basis of personal and professional factors involved.

2. No additional compensation is being provided as a result of heavier teaching loads during each term.

3. Comparatively few faculty members are being assigned exclusively to either the civilian or the training programs.

4. Every effort is being made to avoid any differentiation between salaries for teachers of trainees as compared to salaries for those who teach civilian students.

5. New appointments to the faculty are being made in so far as possible on the same basis as the regular scale of instructional salaries, although a few exceptions have been made in order to procure individuals of special and essential abilities.

6. Institutions generally have resisted any substantial increases in the teaching load, and where such increases have been necessary, every effort has been made to equalize them among all members of the teaching staff. Five institutions did report, however, that the teaching load for all members of the faculty had been increased to twenty hours, the figure originally urged by representatives of the Army and Navy contracting agencies.

7. Institutions are refraining from reducing salaries of those teaching courses with small enrollments due to shifting war demands. They have attempted to equalize total institutional responsibilities by shifting non-teaching activities to those with small instructional loads.

The Faculty Association of the Utah State Agricultural College considered the information made available by institutions replying to a questionnaire which had been widely distributed. The study was not exhaustive, but certain general trends were apparent in the data submitted. A summary follows.

In general, the number of students instructed is between 21 and 30 per section, with a few departments having classes numbering between 31 and 40. The normal peacetime load of English teachers appears to be about 12 hours; for physics and mathematics teachers, 15 hours appears to be the usual load, although there is considerable variation. At the present time, the teaching load of English teachers averages two or three hours more than normal, and the same increase seems to hold true for physics and mathematics instructors. The amount of money allowed for assistance in grading papers, and so on, varies considerably, with approximately 45% of all instructors receiving no assistance for their service courses, approximately 15% receiving assistance as needed, and the remainder receiving allotments with too great a variation to permit the formulation of a definite statement. In general, however, physics departments receive more funds for assistance than either English or mathematics departments. Most schools are attempting to keep each faculty member's teaching load at essentially a normal level, thus avoiding the problem of extra compensation; a few schools pay additional salary for extra hours which must be taught during the regular sessions. Institutions with an academic year of nine months pay additional salary to those who teach during the three intervening months. Thirty-seven per cent of the schools pay at the usual summer session rate during the summer school period. Most of the remainder have a salary scale above the usual summer session stipend. For teaching military personnel between the close of the summer session and the opening of the fall term, 80% grant extra compensation, usually at the regular monthly contractural rate.

EDUCATIONAL CREDIT FOR MILITARY EXPERIENCE

Considerable interest is being expressed by educators relative to plans for the educational placement of veterans returning from the present war. The importance of the problem follows partly from the fact that the Thomas Bill (S. 1509), or some similar measure, is quite certain to be passed by Congress; this Bill provides that many ex-service men possessing the necessary aptitudes and interests "shall be entitled to training at an approved educational or training institution for a period of one year, or for such lesser time as may be required to complete the course of instruction chosen by them. A further period of instruction not exceeding three additional years may be provided for persons of exceptional ability and skill." The levels at which this training must be given are indicated by Army statistics which show that 5.2% of all men in the Army are college graduates, 10.4% attended college but did not graduate, 26.1% received no education past high school graduation, 25.2% had some work in high school, and 33.1% had only grade-schooling or no formal education.

The American Council on Education, with the approval of regional accrediting associations and of the several committees concerned with programs of education in the armed forces, has sponsored a three-point approach to questions of educational credit and placement. The three points proposed as a basis for action are:

(1) *Credit for Military Training.* Institutions which allow credit for ROTC, physical training, hygiene, or free electives may well consider granting direct credit, in these terms, for military training, in proportion to the length and extent of military service and without further examination; not to exceed, however, the total amount of credit available in these fields. (For most individuals this will presumably not exceed one-half semester college credit or one semester high school credit.)

(2) *General Educational Placement.* At the time the individual is discharged from the armed forces, the United States Armed Forces Institute will make available to the educational institutions a "competence profile" of the returning service man (or woman), including his full military and previous educational record and also his Army Classification score and his scores in a battery of tests of general educational competence, to enable the school or college to effect an appropriate educational placement of the applicant in terms of his indicated educational maturity and the extent to which he has met the general educational requirements of the school or college.

(3) *Credit in Special Fields.* On the basis both of his general record and his achievement in the competence level battery, the returning student will be tested further in fields of special competence or training; and upon the scores of these detailed tests, which will also be supplied by the Armed Forces Institute, the receiving institution will be able to determine in terms of its own curricula what specific credit the candidate is entitled to receive in any special fields or subjects; and, as well, to plan and recommend a program of study for him.

In order to carry out such a program, testing materials designed to determine the educational equivalence of military experience are now under construction by a staff of specialists at the University of Chicago. It is planned to carefully verify, validate, and calibrate these materials in actual performance; they are to be reviewed by consultants nominated by the leading professional associations in the respective fields; and they are subject to final certification by the Advisory Committee of the Council as satisfactory for use in this program. In keeping with the type of experience to be evaluated, every effort is being made to construct the tests so that they will give a genuinely educated person an opportunity to demonstrate his competence, regardless of the manner in which it may have been acquired.

The following questions have been taken from a publication of the American Council on Education, and will serve to illustrate the types of materials and techniques employed. The items chosen are taken from that part of the general battery, previously mentioned, which is intended to measure the individual's general mathematical proficiency, at the high school level. Four sample items are given; a full test would include more than 50 items.

1. The current auditor's report shows the assessed valuation of Hillville to run around $1,000,000. Last year the public schools of Hillville spent approximately $25,000. This year $20,000 was requested to cover school costs. What was the reduction in the millage levied for the schools this year as compared with last year?

1) 5 mills.
2) 10 mills.
3) 50 mills.
4) 100 mills.
5) Not given.

2. The approximate volume of a high round-top haystack may be determined by the following formula:

$$V - (.52M - .44W)WL.$$

In this formula W and L represent the stack's width and length. M is the "over" measurement obtained by throwing a rope over the stack and measuring the distance over the stack from a point on the ground on one side of the stack to the corresponding point on the ground on the opposite side. A stack of alfalfa which is 4 months old has an average width of 20 ft. and is 40 ft. long. Its "over" measurement is 40 ft. What is the approximate number of tons of alfalfa in the stack if alfalfa that has settled for more than 90 days runs around 480 cu. ft. per ton?

1) 20.
2) 30.
3) 40.
4) 50.
5) 60.

3. Barrows bought two shares of stock at the same time, the par value of each being $100. For one of the shares he paid $70, and for the other he paid $120. After he had had the stocks for a year, the company for whose stock he paid $120 declared a 6% dividend, while the other company declared a 4% dividend. Which of the following statements about the comparative value of these two investments during the year concerned is true?

1) The actual rate of return is higher on the $70 investment.
2) The actual rate of return is 2% higher on the $120 investment.
3) The actual rates of return on these two investments are exactly equal.
4) The actual rate of return is between 0% and 2% higher on the $120 investment.
5) The actual rate of return is over 2% higher on the $120 investment.

4. Jones operates a sandwich stand. His sandwiches all sell for 10 cents. He also sells such things as soft drinks, coffee, candy bars, homemade soups, pie, cookies, etc. These are sold for either 5 cents or 10 cents. His typical sale is 15 cents, which necessitates his collecting a penny tax. He has plenty of dollar bills in his change drawer but is practically out of all coins. He sends his helper to the bank with a $5 bill for change but neglects to tell what change to get. With which of the following combinations do you believe the helper should return?

1) 10 pennies, 21 nickels, 26 dimes, 5 quarters.
2) 125 pennies, 35 nickels, 20 dimes, 4 quarters.
3) 30 pennies, 20 nickels, 37 dimes.
4) 50 pennies, 10 nickels, 10 dimes, 6 quarters, 3 halves.
5) None of the above is better than any of the others.

FUTURE OF THE NAVY V-12 PROGRAM

The latest word from the Navy Department would indicate that no changes are contemplated in the V-12 Program other than those projected at the initiation of the project. This will probably be only that of anticipated attrition and a slight decrease in the number of basics admitted in the July first cycle.

The comparative stability of the V-12 Program has been partly explained by Dean J. W. Barker, Assistant to the Secretary of the Navy, in the words, "The Navy building program, of course, is a long-range program and we have the necessary places for those who will satisfactorily complete these programs on the commissioning schedule as estimated by us for the ships which are already projected and laid down on the ways. Consequently, our situation is slightly different from that of the Army in that our building program does have to be projected much into the future and our training program can be more closely tied to the actual construction program."*

NOTES ON THE TRAINING PROGRAMS

1. All applicants (A-12 and V-12) taking the two-hour "Army-Navy Qualifying Test" are examined over (a) the meaning and use of words, (b) scientific

matters of general knowledge, and (c) problems in mathematics.

2. "Men with inadequate preparation in mathematics and physics have represented the great bulk of the academic failures in the A.S.T. units."*—Lt. Col. T. D. Palmer, Jr., *Deputy Director, A.S.T.D.*

3. "The high schools of the country have been notified of (the difficulties encountered by V-12 students in their study of mathematics) and they have reported that they have been giving increased attention to the mathematics courses in the senior year in the high schools for the men preparing for these examinations."*—Dean J. W. Barker, *Assistant to the Secretary of the Navy.*

4. The Civilian Pilot Training Act of 1939 elapses on June 30, 1944. Through this legislation the Civil Aeronautics Administration War Training Service Program has been carried on in approximately 300 colleges. Many mathematicians have been employed as instructors. At a recent meeting of the "First National Clinic of Domestic Aviation Planning" held in Oklahoma City a resolution was passed, urging that it is to the national interest to preserve the existing organization of colleges and flight operators conducting Civil Aeronautics Administration War Training Service programs, and that the program should be continued after the cessation of hostilities as an Air Reserve Training Corps.

SELECTIVE SERVICE REGULATIONS CONCERNING STUDENTS

The important Selective Service Local Board Memorandum No. 115 entitled "Occupational Classification Other Than Agricultural" was amended April 4, 1944. The amendment involved special provisions applicable to registrants of ages 18 through 25; these are quoted below.

"Effective immediately, no registrant (whether a nonfather or a father) ages 18 through 25 (other than a registrant who has been found disqualified for any military service or who has been found qualified for limited service only) may be considered as a 'necessary men' entitled to be placed or retained in Class II-A or Class II-B unless: (a) There is filed with the local board a Form 42-A Special upon which the State Director of Selective Service in whose State the registrant's principal place of employment is located has endorsed a statement that, based upon the information in the Form 42-A Special, he recommends that the local board except the registrant from the general restriction against the occupational deferment of registrants ages 18 through 25. (b) An exception to the restriction against occupational deferment of registrants ages 18 through 25 is specifically authorized by the Director of Selective Service without a statement from a State Director of Selective Service, as provided in (a) above, and the local board determines that the registrant comes within the exception described by the Director of Selective Service."

Previous issues of Local Board Memorandum No. 115 have included in the list of exceptions, referred to in (b) above, "students who qualify for occupational deferment in accordance with the provisions of Activity and Occupational

* S.P.E.E. Conference, Chicago, Ill., Oct. 25, 1943.

Bulletin No. 33–6." Since the new Memorandum did not state such an exception, it was implied that students would now be subject to the provisions of the new regulation. Due to the uncertainty involved, however, Leonard Carmichael, Director of the National Roster, sent a clarifying telegram on April 8 to presidents of colleges and universities. His message stated, "Regret to advise that deferment of college students in fields of engineering, physics, chemistry, geology and geophysics under quota system has been rescinded by Selective Service Local Board Memorandum 115 issued in revised form April 4, 1944. Question of continued deferments for students who can graduate before July 1, 1944 and for pre-professional students in medicine and theology now being reconsidered. Will advise you in full detail as soon as final decisions become known to us."

On April 10, a Selective Service announcement sent to state directors said that temporary deferments are recommended for registrants in recognized colleges or universities who will graduate before July 1 and who are pursuing full-time courses in certain scientific and specialized fields. Mathematics was in the list of 24 specialized fields.

THE MATHEMATICAL ASSOCIATION OF AMERICA

NEW MEMBERS

The following forty-two persons have been elected to membership on applications duly certified:

CLAIRE F. ADLER, Ph.D. (New York Univ.) Asst. Prof., New York Univ., New York, N. Y.

K. J. ARNOLD, Ph.D. (Mass. Inst. of Tech.) Asst. Prof., Univ. of New Hampshire, Durham, N. H.

MRS. L. H. BAKER, JR., A.B. (Texas Christian) Jr. Chem. Engr., Humble Oil and Refining Co., Houston, Tex.

C. R. BLYTH. Student, Queen's Univ., Kingston, Ont., Can.

J. D. BURK, B.A. (Toronto) Asso. Prof., Univ. of Toronto, Toronto, Ont., Can.

L. VIRGINIA CARLTON, A.M. (Tulane) Asst. Prof., Wesleyan Coll., Macon, Ga.

K. L. CLARK, Ph.B. (Wisconsin) Mathematician, A. O. Smith Corp., Milwaukee, Wis.

L. C. DAMSGARD, A.B. (Union Coll., Nebr.) Asst. Prof., Math. and Astr., Pasadena Jr. Coll., Pasadena, Calif.

J. E. EATON, Ph.D. (Yale) Instr., Queens Coll., Flushing, N.Y.

WILL FELLER, Ph.D. (Göttingen) Asso. Prof., Brown Univ., Providence, R. I.

M. J. FORRAY, A.B. (New York Univ.) Instr., Washington Square Coll., New York Univ., New York, N. Y.

MARY A. GOINS, A.M. (Michigan) Asso. Prof., Western Coll., Oxford, Ohio

A. W. GOODMAN, A.M. (Cincinnati) Instr., Syracuse Univ., Syracuse, N. Y.

MRS. LEOTA C. HAYWARD, M.S. (Colorado State Coll.) Asst. Prof., Colorado State Coll. of A. and M. A., Fort Collins, Colo.

FRITZ HERZOG, Ph.D. (Columbia) Asst. Prof., Michigan State Coll., East Lansing, Mich.

N. L. JACOBSON, B.S. in Educ. (Oregon) Instr., Central Missouri State Teachers Coll., Warrensburg, Mo.

MAX KATZ. Accountant, Katz, Zuckerman and Co., New York, N. Y.

F. C. LEONE, M.S. (Georgetown Univ.) Instr., Purdue Univ., West Lafayette, Ind.

L. D. LEVINE, A.B. (Brooklyn) Instr., Carnegie Inst. of Tech., Pittsburgh, Pa.

LIONEL LONDON, B.S.C.E. (Illinois) Analyst, A. O. Smith Corp., Milwaukee, Wis.

S. S. MCNEARY, A.M. (Pennsylvania) Instr., Drexel Inst., Philadelphia, Pa.

W. B. MICHAEL, A.B. (U.C.L.A.) Instr., Pasadena Jr. Coll., Pasadena, Calif.

A. C. MOELLER, B.S. (Western Reserve) Instr., Denison Univ., Granville, Ohio

C. E. MULLAN, A.M. (Duquesne) Senior Engr., Duquesne Light Co., Pittsburgh, Pa.

P. F. NEMÉNYI, D.Sc. (Berlin) Acting Asst. Prof., State Coll. of Washington, Pullman, Wash.

J. C. OXTOBY, A.M. (California) Asso. Prof., Bryn Mawr Coll., Bryn Mawr, Pa.

BOB PARKER, A.M. (Texas Tech.) Instr., Texas Tech. Coll., Lubbock, Tex.

WALTER PRENOWITZ, Ph.D. (Columbia) Asst. Prof., Brooklyn Coll., Brooklyn, N. Y.

L. W. RAMSEY, B.S. (A. and M. Coll. of Texas) Asst. Prof., Math. and Engr., Texas Christian Univ., Fort Worth, Tex.

R. M. ROBINSON, Ph.D. (California) Asst. Prof., Univ. of California, Berkeley, Calif.

MARGARITA RODRÍGUEZ. Aux. Prof., Matanzas Inst., Havana, Cuba.

G. F. ROSE, B.S. (Case School) Research Engr., Westinghouse Elec. and Mfg. Co., East Pittsburgh, Pa.

G. S. ROSENBERG. Student, Illinois Inst. of Tech., Chicago, Ill.

M. J. SHEEHY, A.B. (U.C.L.A.) Asst. Physicist, Univ. of California Div. of War Research, U. S. Navy Radio and Sound Lab., San Diego, Calif.

R. H. SORGENFREY, Ph.D. (Texas) Instr., Univ. of California at Los Angeles, Los Angeles, Calif.

HILLEL SPITZ, A.B. (George Washington) Instr., Univ. of New Hampshire, Durham, N. H.

E. L. STANLEY, M.S. (Tennessee) Instr., Clemson Coll., Clemson, S. C.

L. W. STARK, A.B. (Univ. of Kansas City) Acting Prof., Acting Head of Dept., William Jewell Coll., Liberty, Mo.

J. S. STUBBE, M.S. (Brown) Instr., Univ. of Cincinnati, Cincinnati, Ohio

J. W. SWANK. Statistician, Southern California Edison Co., Ltd., Los Angeles, Calif.

F. A. VALENTINE, Ph.D. (Chicago) Asst. Prof., Univ. of California at Los Angeles, Los Angeles, Calif.

H. S. ZUCKERMAN, Ph.D. (California) Asst. Prof., Univ. of Washington, Seattle, Wash.

THE SIXTH ANNUAL MEETING OF THE NORTHERN CALIFORNIA SECTION

The sixth annual meeting of the Northern California Section of the Mathematical Association of America was held at the University of California at Berkeley, California, on Saturday, January 29, 1944. In the absence of Professor E. B. Roessler, Chairman of the Section, the Vice-Chairman, Professor Gabor Szegö, presided at the morning session. Professors Sophia H. Levy and F. R. Morris presided at the afternoon session. During the noon recess luncheon was served at the Men's Faculty Club.

The meeting was attended by sixty persons, including the following twenty-three members of the Association: H. M. Bacon, G. A. Baker, H. W. Becker, B. A. Bernstein, G. C. Evans, S. A. Francis, F. T. Frank, Emma V. Hesse, D. H. Lehmer, Sophia H. Levy, E. D. Miller, F. R. Morris, W. H. Myers, E. J. Phillips, R. M. Robinson, Ethel Spearman, Pauline Sperry, Ruth G. Sumner, Gabor Szegö, R. K. Wakerling, A. R. Williams, Frantisek Wolf, Clyde Wolfe.

The following officers were elected for the coming year: Chairman, Gabor Szegö, Stanford University; Vice-Chairman, Pauline Sperry, University of California; Secretary-Treasurer, H. M. Bacon, Stanford University. Mrs. Ruth G. Sumner, Oakland High School, was reelected to represent the Section as associate editor of the *California Journal of Secondary Education.*

The following program was presented:

1. *Euclidian metric invariants of conics by tensor algebra*, by Dr. T. C. Doyle, Stanford University, introduced by the Secretary.

The speaker illustrated the application of tensor algebra to the invariant theory of the conics and the reduction of their equations to canonical form. This was done by extending the euclidian metric group

$$x^i = a^i_r x^r + a^i_0, \qquad (i = 1, 2),$$

to contain one more variable x^0, and then applying this group to the general equation

$$g_{rs}x^r x^s + g_{os}x^s + g_{oo} = 0$$

of the conic after a similar extension to three variables.

2. *Solutions of several problems in heat transfer*, by Professor L. M. K. Boelter, University of California, introduced by the Chairman.

Professor Boelter, who is Associate Dean of the College of Engineering, University of California, appeared on the program by invitation of the Section. He spoke for about an hour on certain topics relating to mathematics in engineering. He discussed three methods by which engineering problems are solved, namely: (1) application of criteria based upon accumulated experience; (2) extrapolation of data from model studies, wherein physical (as well as geometric) conditions of similarity have been met; (3) analytical or graphical solution of differential equations defining idealized systems which resemble, but never coin-

cide with, the real systems under investigation. The considerations which determine the choice of method were outlined. It was also remarked that the engineering student should be encouraged to become familiar with certain phases of mathematics such as differential equations, probability, statistics, and some of the less familiar functions which occur in the analysis of physical systems.

3. *Present trends in secondary school mathematics*, by Professor L. B. Kinney, Acting Dean of the School of Education, Stanford University, introduced by the Vice-Chairman.

Dean Kinney outlined those trends which have become apparent in the adjustments of the schools to the present emergency. He also commented upon the programs in the armed forces and in industry. It was remarked that the perceptible trends result in the main from the emphasis upon mere ability to perform mathematical operations, and from the need for mathematical competence in a large group of individuals previously illiterate mathematically. The following trends were noted: (1) an increasing emphasis upon mathematics for use; (2) the tendency for the organization of courses to emphasize the social and personal significance of the subject, as well as the systematic nature of the field; (3) the programs set up for special groups of students tend to be liberated from dominance by the general student; (4) there is increased utilization of the more varied approaches and activities in the mathematics classroom; (5) there are fewer topics in the typical mathematics course, but the topics included are more completely developed; (6) there promises to be a continued demand for mathematics as a part of a general education.

4. *Some problems connected with discordant permutations*, by Professor D. H. Lehmer, University of California.

The speaker discussed some recent advances in connection with several problems relating to discordant permutations. Among the topics discussed were the recently published results of Schöbe and Kaplansky on the *problèm des ménages*, the contribution of Narashimamurti to the problem of cable splicing, and the results of Kariwala, Riordan, and Becker on Latin configurations.

5. *An application of conformal mapping to aeronautical engineering*, by H. J. Shaw, Stanford University, introduced by the Secretary.

This paper was a summary of an application of conformal mapping which was given by T. Theodorsen and I. E. Garrick in their paper on *General Potential Theory of Arbitrary Wing Section* in the N. A. C. A. *Bulletin* for 1933. The problem considered was the determination of the operating characteristics of an airplane wing having a cross section of arbitrary shape. It was first noted that the problem can be reduced to that of mapping the boundary of the wing cross section conformally upon the boundary of a circle. The method of mapping a graphically given wing section of arbitrary shape is as follows. The wing section is subjected to the transformation $w=z+(1/z)$. This maps any practical wing section in the w plane upon a nearly circular curve S in the z plane. It can be

shown that if the points on the boundary of S are denoted by $e^{\Phi+i\Psi}$ and points on the circle by $e^{i\psi}$, then the mapping of the near circle upon the circle is given by the integral equation

$$\Psi = \psi + \frac{1}{2\pi}\int_0^{2\pi} \Phi(\psi) \cot \frac{t-\psi}{2}\, dt.$$

Numerical solutions of this equation for a number of points on the boundaries can be obtained by successive approximations. The process converges so rapidly that in practice the first approximation is usually sufficient.

6. *Graphical methods in exterior ballistics*, by Dr. L. H. Swinford, University of California, introduced by the Chairman.

In this address it was pointed out that the integration of the classical system of differential equations for the solution of the special problem of exterior ballistics cannot be performed by quadratures for an arbitrary or empirical resistance law. It was noted that the method of integration by successive approximations is one of the favored methods, and that this can well be carried out graphically. The graphical method may be preferable even in cases in which the principal equation is integrable by quadratures, since these quadratures are awkward. It was also remarked that the graphical method would seem to be preferable to the Euler-Otto method for the quadratic law of resistance.

7. *Solution of cubic and quartic equations by the method of identities*, by V. F. Ivanoff, San Francisco.

This paper was read by title.

H. M. BACON, *Secretary*

CALENDAR OF FUTURE MEETINGS

Twenty-Seventh Summer Meeting, Wellesley, Mass., August 12–14, 1944.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENEY MOUNTAIN
ILLINOIS
INDIANA, Indianapolis, November 10, 1944
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA, St. Paul, May 6, 1944
MISSOURI
NEBRASKA, Lincoln, May 6, 1944
NORTHERN CALIFORNIA, San Francisco, January 27, 1945
OHIO
OKLAHOMA
PHILADELPHIA, Philadelphia, November, 1944
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Los Angeles, March 10, 1945
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN, Milwaukee, May, 1944

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